Spin Precession in Electromagnetic Field

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1 T-BMT Equation

This note is an explicit calculation of the spin precession known as T-BMT equation. The method used in this note is based on `arXiv:1308.1580` and Jackson, *Classical Electrodynamics* (3rd edition) with SI unit. In the rest frame of a particle, the time evolution of spin 3-vector ($\mathbf{s}$) in electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields can be written as

$$\frac{d\mathbf{s}}{dt} = 2\mu(\mathbf{s} \times \mathbf{B}) + 2d(\mathbf{s} \times \mathbf{E}) \tag{1}$$

where μ and d are magnetic and electric dipole moments, respectively (MDM: magnetic dipole moment, EDM: electric dipole moment). Now, let us discuss how it looks like when it is written in relativistically covariant form.

In a reference frame, [1] let us define the momentum and spin 4-vector of a relativistic particle as S^μ and P^μ, respectively. In the frame that the particle is at rest, we have

$$s^\mu = (0, \mathbf{s}), \quad p^\nu = (mc, \mathbf{0}) \tag{2}$$

where c is the speed of light. Note that $\mathbf{s}$ is the spin 3-vector in the rest frame of the particle. Then, Lorentz invariance required that

$$S^\mu P_\mu = s^\mu p_\mu = 0, \quad S^\mu S_\mu = s^\mu s_\mu = -\mathbf{s} \cdot \mathbf{s} = -\mathbf{s}^2 \tag{3}$$

[1] In this note, capital-lettered 4-vectors are defined in the lab frame, and lowercased 4-vectors are defined in the rest frame of the particle. Also, we use sign convention of $S^\mu = (S^0, \mathbf{S})$, $S_\mu = (S^0, -\mathbf{S})$.

are satisfied. Also, if we use velocity 4-vector $U^\mu = \gamma(c, \mathbf{v})$,

$$S^\mu U_\mu = s^\mu u_\mu = 0, \quad S_0 = \boldsymbol{\beta} \cdot \mathbf{S} \tag{4}$$

is satisfied. ($\boldsymbol{\beta} = \mathbf{v}/c, \gamma = 1/\sqrt{1-\beta^2}$).

Now, the spin 4-vector (S^μ) can be obtained by Lorentz transformation on s^μ just like the case for the 4-momentum. According to the notation in Jackson (p. 525), Lorentz transformation of space-time can be expressed as

$$\begin{aligned} x_0' &= \gamma(x_0 - \boldsymbol{\beta} \cdot \mathbf{x}), \\ \mathbf{x}' &= \mathbf{x} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{x})\boldsymbol{\beta} - \gamma\boldsymbol{\beta} x_0, \end{aligned} \tag{5}$$

when two frames move with the relative velocity $\boldsymbol{\beta}$. To apply them to our spin case, let us assume that ($'$)-system is the rest frame of the particle. Then with $S_0 = \boldsymbol{\beta} \cdot \mathbf{S}$,

$$\begin{aligned} \mathbf{s} &= \mathbf{S} + \frac{\gamma - 1}{\beta^2}(\boldsymbol{\beta} \cdot \mathbf{S})\boldsymbol{\beta} - \gamma\boldsymbol{\beta} S_0 \\ &= \mathbf{S} + \left[\frac{\gamma - 1}{\beta^2} - \gamma\right](\boldsymbol{\beta} \cdot \mathbf{S})\boldsymbol{\beta} \\ &= \mathbf{S} - \frac{\gamma}{\gamma + 1}(\boldsymbol{\beta} \cdot \mathbf{S})\boldsymbol{\beta} \end{aligned} \tag{6}$$

is obtained. Since we are interested in the inverse of them, let's use $S^0 = \gamma(s^0 + \boldsymbol{\beta} \cdot \mathbf{s})$, $\therefore S_0 = S^0 = \gamma\boldsymbol{\beta} \cdot \mathbf{s}$, so $\boldsymbol{\beta} \cdot \mathbf{S} = S_0 = \gamma\boldsymbol{\beta} \cdot \mathbf{s}$ can be obtained. Now, the reverse transformation is

$$\begin{aligned} \mathbf{S} &= \mathbf{s} + \frac{\gamma}{\gamma + 1}(\boldsymbol{\beta} \cdot \mathbf{S})\boldsymbol{\beta} \\ &= \mathbf{s} + \frac{\gamma^2}{\gamma + 1}(\boldsymbol{\beta} \cdot \mathbf{s})\boldsymbol{\beta}. \end{aligned} \tag{7}$$

Now let us construct the relativistic equation of motion for spin. The left-hand side of such equation should be the Lorentz covariant form of $dS^\mu/d\tau$. The right-hand side may contain electromagnetic field, spin, or velocity 4-vector. If we can neglect terms like $\nabla(\boldsymbol{\mu} \cdot \mathbf{B})$, we may write the spin equation as (see arguments developed in Jackson, p.

563. In principle we can have terms like $U^\mu \partial_\nu F^{\nu\lambda} S_\lambda$?)

$$\frac{dS^\mu}{d\tau} = c_1 F^{\mu\nu} S_\nu + c_2 U^\mu F^{\nu\lambda} U_\nu S_\lambda + c_3 F^{*\mu\nu} S_\nu + c_4 U^\mu F^{*\nu\lambda} U_\nu S_\lambda \tag{8}$$

where c_i $(i = 1, 2, 3, 4)$ are constants to be determined later and

$$\begin{aligned} F^{\mu\nu} &= \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}, \\ F^{*\mu\nu} &= \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{pmatrix}. \end{aligned} \tag{9}$$

Here $F^{\mu\nu}$ is the field-strength tensor (Jackson p. 556) and $F^{*\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. Now, the spatial components of Eq. (8) at the particle rest frame are

$$\frac{ds^i}{dt} = c_1 F^{ij} s_j + c_3 F^{*ij} s_j \tag{10}$$

and using $s^\mu = (0, \mathbf{s}), s_\mu = (0, -\mathbf{s})$, one can write them as

$$\begin{aligned} F^{1j} S_j &= (B_z s_y - B_y s_z) = (\mathbf{s} \times \mathbf{B})_x \\ F^{2j} S_j &= (-B_z s_x + B_x s_z) = (\mathbf{s} \times \mathbf{B})_y \\ F^{3j} S_j &= (B_y s_x - B_x s_y) = (\mathbf{s} \times \mathbf{B})_z \\ F^{*1j} S_j &= (-E_z s_y + E_y s_z)/c = (\mathbf{E} \times \mathbf{s})_x/c \\ F^{*2j} S_j &= (E_z s_x - E_x s_z)/c = (\mathbf{E} \times \mathbf{s})_y/c \\ F^{*3j} S_j &= (-E_y s_x + E_x s_y)/c = (\mathbf{E} \times \mathbf{s})_z/c. \end{aligned} \tag{11}$$

Therefore,

$$\frac{ds^i}{dt} = c_1 (\mathbf{s} \times \mathbf{B})^i + c_3 (\mathbf{E} \times \mathbf{s})^i / c \tag{12}$$

is satisfied. This should be consistent with Eq. (1) so we conclude that

$$c_1 = 2\mu, \quad c_3 = -2dc \tag{13}$$

must be satisfied. On ther other hand, the value of c_2 can be obtained from the relativistic equation of motion $m\frac{dU^\mu}{d\tau} = eF^{\mu\nu}U_\nu$ and $S_\mu U^\mu = 0$. If we differentiate $S_\mu U^\mu = 0$ with respect to the proper time,

$$U_\mu \frac{dS^\mu}{d\tau} = -S_\mu \frac{dU^\mu}{d\tau} = -S_\mu \frac{e}{m} F^{\mu\nu} U_\nu = \frac{e}{m} F^{\mu\nu} U_\mu S_\nu \tag{14}$$

is satisfied where we use anti-symmetric nature of $F^{\mu\nu}$ with indices μ, ν at the last step. Now, contract the velocity 4-vector U_μ to Eq. (8) and use $U^\mu U_\mu = c^2$, one gets

$$\begin{aligned} U_\mu \frac{dS^\mu}{d\tau} &= 2\mu F^{\mu\nu} U_\mu S_\nu + c_2 U_\mu U^\mu F^{\nu\lambda} U_\nu S_\lambda - 2dc F^{*\mu\nu} U_\mu S_\nu + c_4 U_\mu U^\mu F^{*\nu\lambda} U_\nu S_\lambda \\ &= (2\mu + c^2 c_2) F^{\mu\nu} U_\mu S_\nu + (-2dc + c^2 c_4) F^{*\mu\nu} U_\mu S_\nu \\ &= \frac{e}{m} F^{\mu\nu} U_\mu S_\nu \end{aligned} \tag{15}$$

so

$$\Big(2\mu + c^2 c_2 - \frac{e}{m}\Big) F^{\mu\nu} U_\mu S_\nu + (-2dc + c^2 c_4) F^{*\mu\nu} U_\mu S_\nu = 0 \tag{16}$$

must be satisfied. Therefore,

$$c_2 = -\frac{2}{c^2}\Big(\mu - \frac{e}{2m}\Big), \quad c_4 = \frac{2d}{c} \tag{17}$$

are found. The spin 4-vector equation of motion is now

$$\frac{dS^\mu}{d\tau} = 2\mu F^{\mu\nu} S_\nu - \frac{2}{c^2}\Big(\mu - \frac{e}{2m}\Big) U^\mu F^{\nu\lambda} U_\nu S_\lambda - 2cd(F^{*\mu\nu} S_\nu - \frac{1}{c^2} U^\mu F^{*\nu\lambda} U_\nu S_\lambda) \tag{18}$$

and Eq. (18) is known as the relativistic Thomas-Bargmann-Michel-Telegdi (T-BMT) equation, or the spin 4-vector equation of motion.

Now let us look at the spatial components of $ds^\mu/d\tau$. Before we do this, let us define

$\mu' \equiv \mu - e/2m$ and rewrite the T-BMT equation as

$$\frac{dS^\mu}{d\tau} = 2\mu F^{\mu\nu} S_\nu - \frac{2}{c^2}\mu' U^\mu F^{\nu\lambda} U_\nu S_\lambda - 2cd(F^{*\mu\nu} S_\nu - \frac{1}{c^2} U^\mu F^{*\nu\lambda} U_\nu S_\lambda). \tag{19}$$

To further simplify the discussion, let us write the MDM and EDM parts separately as

$$\frac{d\mathbf{S}}{dt} = \left[\frac{d\mathbf{S}}{dt}\right]_{\mathrm{MDM}} + \left[\frac{d\mathbf{S}}{dt}\right]_{\mathrm{EDM}}. \tag{20}$$

Then

$$\gamma \left[\frac{dS^1}{dt}\right]_{\mathrm{MDM}} = 2\mu F^{1\nu} S_\nu - \frac{2}{c^2}\mu' U^1 F^{\nu\lambda} U_\nu S_\lambda \tag{21}$$

and the first term $F^{1\nu} S_\nu$ becomes

$$\begin{aligned} F^{1\nu} S_\nu &= F^{10} S_0 + F^{12} S_2 + F^{13} S_3 = (E_x/c)(\boldsymbol{\beta} \cdot \mathbf{S}) + (-B_z)(-S_y) + (B_y)(-S_z) \\ &= (E_x/c)(\boldsymbol{\beta} \cdot \mathbf{S}) + (\mathbf{S} \times \mathbf{B})_x \end{aligned} \tag{22}$$

so in general

$$F^{i\nu} S_\nu = (\boldsymbol{\beta} \cdot \mathbf{S}) \frac{\mathbf{E}^i}{c} + (\mathbf{S} \times \mathbf{B})^i \tag{23}$$

is satisfied. Now, the second term $U^1F^{\nu\lambda}U_\nu S_\lambda$ becomes

$$\begin{aligned}
U^1F^{\nu\lambda}U_\nu S_\lambda &= U^1(F^{0\lambda}U_0S_\lambda + F^{1\lambda}U_1S_\lambda + F^{2\lambda}U_2S_\lambda + F^{3\lambda}U_3S_\lambda) \\
&= U^1(F^{01}U_0S_1 + F^{02}U_0S_2 + F^{03}U_0S_3) \\
&+ U^1(F^{10}U_1S_0 + F^{12}U_1S_2 + F^{13}U_1S_3) \\
&+ U^1(F^{20}U_2S_0 + F^{21}U_2S_1 + F^{23}U_2S_3) \\
&+ U^1(F^{30}U_3S_0 + F^{31}U_3S_1 + F^{32}U_3S_2) \\
&= U^1(\gamma c)\Big((-E_x/c)(-S_x) + (-E_y/c)(-S_y) + (-E_z/c)(-S_z)\Big) \\
&+ U^1(\gamma)\Big((E_x/c)(-v_x)(S_0) + (-B_z)(-v_x)(-S_y) + (B_y)(v_x)(-S_z)\Big) \\
&+ U^1(\gamma)\Big((E_y/c)(-v_y)(S_0) + (B_z)(-v_y)(-S_x) + (-B_x)(v_y)(-S_z)\Big) \\
&+ U^1(\gamma)\Big((E_z/c)(-v_z)(S_0) + (-B_y)(-v_z)(-S_x) + (B_x)(v_z)(-S_y)\Big) \\
&= U^1\gamma\Big(\mathbf{E}\cdot\mathbf{S} - (\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{S}) - c\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\Big). \qquad (24)
\end{aligned}$$

So,

$$\begin{aligned}
U^iF^{\nu\lambda}U_\nu S_\lambda &= U^i\gamma\Big(\mathbf{E}\cdot\mathbf{S} - (\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{S}) - c\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\Big) \\
&= c\boldsymbol{\beta}^i\gamma^2\Big(\mathbf{E}\cdot\mathbf{S} - (\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{S}) - c\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\Big). \qquad (25)
\end{aligned}$$

Therefore,

$$\left[\frac{d\mathbf{S}}{dt}\right]_{\mathrm{MDM}} = \frac{2}{\gamma}\left[\mu\Big\{\mathbf{S}\times\mathbf{B} + (\mathbf{S}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}\Big\} + \mu'\gamma^2\frac{\boldsymbol{\beta}}{c}\Big\{-\mathbf{S}\cdot\mathbf{E} + (\boldsymbol{\beta}\cdot\mathbf{E})(\boldsymbol{\beta}\cdot\mathbf{S}) + c\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\Big\}\right]$$

is satisfied. Now EDM term can be obtained without such complicated steps by noting that the following transformation

$$F^{\mu\nu} \to F^{*\mu\nu} \qquad (26)$$

can be made by $\mathbf{E}/c \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}/c$. Therefore

$$\begin{aligned} F^{*i\nu}S_\nu &= F^{i\nu}S_\nu\Big|_{\frac{\mathbf{E}}{c}\to\mathbf{B},\ \ \mathbf{B}\to-\frac{\mathbf{E}}{c}} \\ &= \left[\mathbf{S}\times\Big(-\frac{\mathbf{E}}{c}\Big)+(\mathbf{S}\cdot\boldsymbol{\beta})\mathbf{B}\right]^i \end{aligned} \tag{27}$$

and

$$\begin{aligned} U^iF^{*\nu\lambda}U_\nu S_\lambda &= U^iF^{\nu\lambda}U_\nu S_\lambda\Big|_{\frac{\mathbf{E}}{c}\to\mathbf{B},\ \ \mathbf{B}\to-\frac{\mathbf{E}}{c}} \\ &= c\boldsymbol{\beta}^i\gamma^2\left(c\mathbf{B}\cdot\mathbf{S}-(\boldsymbol{\beta}\cdot c\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})-c\boldsymbol{\beta}\cdot\Big(\mathbf{S}\times\Big(-\frac{\mathbf{E}}{c}\Big)\Big)\right) \end{aligned} \tag{28}$$

are satisfied. Therefore

$$\left[\frac{d\mathbf{S}}{dt}\right]_{\text{EDM}} = -\frac{2cd}{\gamma}\left[(\mathbf{S}\cdot\boldsymbol{\beta})\mathbf{B}-\mathbf{S}\times\frac{\mathbf{E}}{c}+\frac{\boldsymbol{\beta}}{c}\gamma^2\Big\{-c\mathbf{S}\cdot\mathbf{B}-\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{E})+c(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})\Big\}\right]$$

can be written. So,

$$\begin{aligned} \left[\frac{d\mathbf{S}}{dt}\right]_{\text{MDM}} &= \frac{2}{\gamma}\left[\mu\Big\{\mathbf{S}\times\mathbf{B}+(\mathbf{S}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}\Big\}+\mu'\gamma^2\boldsymbol{\beta}\Big\{-\mathbf{S}\cdot\frac{\mathbf{E}}{c}+\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)(\boldsymbol{\beta}\cdot\mathbf{S})+\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\Big\}\right] \\ \left[\frac{d\mathbf{S}}{dt}\right]_{\text{EDM}} &= -\frac{2cd}{\gamma}\left[(\mathbf{S}\cdot\boldsymbol{\beta})\mathbf{B}-\mathbf{S}\times\frac{\mathbf{E}}{c}+\boldsymbol{\beta}\gamma^2\Big\{-\mathbf{S}\cdot\mathbf{B}-\boldsymbol{\beta}\cdot\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)+(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})\Big\}\right] \end{aligned}$$

is satisfied (identical to `arXiv:1308.1580` Eq. (15)). The above expressions can be rewritten with

$$\mu=\frac{e}{4m}g,\quad \mu'=\mu-\frac{e}{2m}=\frac{e}{4m}(g-2)=\frac{e}{2m}\frac{g-2}{2}. \tag{29}$$

If we do that,

$$\begin{aligned}\left[\frac{d\mathbf{S}}{dt}\right]_{\text{MDM}} &= \frac{2}{\gamma}\frac{e}{4m}\left[g\Big\{\mathbf{S}\times\mathbf{B}+(\mathbf{S}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}\Big\}\right.\\ &+ \left.(g-2)\gamma^2\boldsymbol{\beta}\Big\{-\mathbf{S}\cdot\frac{\mathbf{E}}{c}+(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c})(\boldsymbol{\beta}\cdot\mathbf{S})+\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\Big\}\right] \qquad (30)\end{aligned}$$

is made. On the other hand, the Lorentz force equation tells us

$$\frac{d\boldsymbol{\beta}}{dt}=\frac{e}{\gamma m}\left[\boldsymbol{\beta}\times\mathbf{B}+\frac{\mathbf{E}}{c}-\boldsymbol{\beta}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)\right] \qquad (31)$$

and

$$\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{dt}=\frac{e}{\gamma m}\left[\boldsymbol{\beta}\cdot(\boldsymbol{\beta}\times\mathbf{B})+\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}-\beta^2\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)\right]=\frac{e}{\gamma m}\left[\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)(1-\beta^2)\right]=\frac{e}{\gamma^3 m}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big).$$

Now,

$$\beta\frac{d\beta}{dt}=\frac{1}{2}\frac{d(\beta^2)}{dt}=\frac{1}{2}\frac{d(\boldsymbol{\beta}\cdot\boldsymbol{\beta})}{dt}=\boldsymbol{\beta}\cdot\frac{d\boldsymbol{\beta}}{dt}=\frac{e}{\gamma^3 m}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big) \qquad (32)$$

and

$$\mathbf{s}=\mathbf{S}-\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta} \qquad (33)$$

give us

$$\begin{aligned}\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM}} &= \left[\frac{d\mathbf{S}}{dt}\right]_{\text{MDM}}-\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\left[\frac{d\mathbf{S}}{dt}\right]_{\text{MDM}}\right)\boldsymbol{\beta}-\frac{\gamma}{\gamma+1}\left(\frac{d\boldsymbol{\beta}}{dt}\cdot\mathbf{S}\right)\boldsymbol{\beta}\\ &- \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{S})\frac{d\boldsymbol{\beta}}{dt}-\frac{\gamma^3}{(\gamma+1)^2}\beta\frac{d\beta}{dt}(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta}\\ &= \left[\frac{d\mathbf{S}}{dt}\right]_{\text{MDM}}-\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\left[\frac{d\mathbf{S}}{dt}\right]_{\text{MDM}}\right)\boldsymbol{\beta}-\frac{\gamma}{\gamma+1}\left(\frac{d\boldsymbol{\beta}}{dt}\cdot\mathbf{S}\right)\boldsymbol{\beta}\\ &- \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{S})\frac{d\boldsymbol{\beta}}{dt}-\frac{1}{(\gamma+1)^2}\frac{e}{m}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta}. \qquad (34)\end{aligned}$$

We want to write above equations with physical quantities defined at the rest frame of the particle. To do that, first we put Eq. (30) and others in the above equation. Then,

$$
\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM}} &= \frac{1}{\gamma}\frac{e}{m}\left[\frac{g}{2}\left\{\mathbf{S}\times\mathbf{B}+(\mathbf{S}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}\right\}\right. \\
&+ \left.\frac{(g-2)}{2}\gamma^2\boldsymbol{\beta}\left\{-\mathbf{S}\cdot\frac{\mathbf{E}}{c}+\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)(\boldsymbol{\beta}\cdot\mathbf{S})+\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\right\}\right] \\
&- \frac{1}{\gamma+1}\frac{e}{m}\left[\frac{g}{2}\left\{\mathbf{S}\times\mathbf{B}+(\mathbf{S}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}\right\}\right. \\
&+ \left.\frac{(g-2)}{2}\gamma^2\boldsymbol{\beta}\left\{-\mathbf{S}\cdot\frac{\mathbf{E}}{c}+\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)(\boldsymbol{\beta}\cdot\mathbf{S})+\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\right\}\right]\cdot\boldsymbol{\beta}\boldsymbol{\beta} \\
&- \frac{1}{\gamma+1}\frac{e}{m}\left[\boldsymbol{\beta}\times\mathbf{B}+\frac{\mathbf{E}}{c}-\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\right]\cdot\mathbf{S}\boldsymbol{\beta} \\
&- \frac{1}{\gamma+1}\frac{e}{m}(\boldsymbol{\beta}\cdot\mathbf{S})\left[\boldsymbol{\beta}\times\mathbf{B}+\frac{\mathbf{E}}{c}-\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\right] \\
&- \frac{1}{(\gamma+1)^2}\frac{e}{m}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta} \qquad (35)
\end{aligned}
$$

and is quite complicated. Now let us use $\boldsymbol{\beta}\cdot\mathbf{S}=\gamma\boldsymbol{\beta}\cdot\mathbf{s}$ and the relation between $\mathbf{S}$ and $\mathbf{s}$. First, let's collect terms with the electric field only. Then

$$
\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM, }\mathbf{E}\text{ term only}} &= \frac{e}{m}\gamma(\mathbf{s}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}\left[\frac{g}{2\gamma}-\frac{1}{\gamma+1}\right] \\
&+ \frac{e}{m}\left(\mathbf{S}\cdot\frac{\mathbf{E}}{c}\right)\boldsymbol{\beta}\left[-\left(\frac{g-2}{2}\right)\gamma+\left(\frac{g-2}{2}\right)\frac{\gamma^2\beta^2}{\gamma+1}-\frac{1}{\gamma+1}\right. \\
&+ \frac{e}{m}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\gamma(\boldsymbol{\beta}\cdot\mathbf{s})\left[\left(\frac{g-2}{2}\right)\gamma-\frac{g}{2}\frac{1}{\gamma+1}-\left(\frac{g-2}{2}\right)\frac{\gamma^2\beta^2}{\gamma+1}\right. \\
&+ \left.\frac{1}{\gamma+1}+\frac{1}{\gamma+1}-\frac{1}{(\gamma+1)^2}\right] \\
&= \frac{e}{m}\left(\frac{g-2}{2}+\frac{1}{\gamma+1}\right)\left[(\mathbf{s}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}-\left(\mathbf{S}\cdot\frac{\mathbf{E}}{c}\right)\boldsymbol{\beta}\right] \\
&+ \frac{e}{m}\boldsymbol{\beta}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\gamma(\boldsymbol{\beta}\cdot\mathbf{s})\left[\frac{\gamma}{\gamma+1}\left\{\frac{g-2}{2}+\frac{1}{\gamma+1}\right\}\right]. \qquad (36)
\end{aligned}
$$

Now, let's use $\mathbf{S} = \mathbf{s} + \gamma^2/(\gamma+1)(\boldsymbol{\beta}\cdot\mathbf{s})\boldsymbol{\beta}$ to rewrite above. Then

$$\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM, }\mathbf{E}\text{ terms only}} &= \frac{e}{m}\Big(\frac{g-2}{2}+\frac{1}{\gamma+1}\Big)\left[(\mathbf{s}\cdot\boldsymbol{\beta})\frac{\mathbf{E}}{c}-\Big(\mathbf{s}\cdot\frac{\mathbf{E}}{c}\Big)\boldsymbol{\beta}-\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{s})\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)\boldsymbol{\beta}\right] \\
&+ \frac{e}{m}\boldsymbol{\beta}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)(\boldsymbol{\beta}\cdot\mathbf{s})\left[\frac{\gamma^2}{\gamma+1}\Big\{\frac{g-2}{2}+\frac{1}{\gamma+1}\Big\}\right] \\
&= \frac{e}{m}\Big(\frac{g-2}{2}+\frac{1}{\gamma+1}\Big)\mathbf{s}\times\Big(\frac{\mathbf{E}}{c}\times\boldsymbol{\beta}\Big). \qquad (37)
\end{aligned}$$

Now let's collect the magnetic field terms. Then

$$\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM, }\mathbf{B}\text{ terms only}} &= \frac{e}{m}\mathbf{S}\times\mathbf{B}\left[\frac{g}{2\gamma}\right] \\
&+ \frac{e}{m}\boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{B})\left[\Big(\frac{g-2}{2}\Big)\gamma-\frac{g}{2}\frac{1}{\gamma+1}-\Big(\frac{g-2}{2}\Big)\frac{\gamma^2\beta^2}{\gamma+1}\right]\boldsymbol{\beta} \\
&+ \frac{e}{m}(\boldsymbol{\beta}\times\mathbf{B})\cdot\mathbf{S}\left[-\frac{1}{\gamma+1}\right]\boldsymbol{\beta}-\frac{e}{m}\frac{1}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{S})(\boldsymbol{\beta}\times\mathbf{B}) \\
&= \frac{e}{m}\frac{g}{2\gamma}\left[\mathbf{s}+\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta}\right]\times\mathbf{B} \\
&+ \frac{e}{m}\Big(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\Big)\boldsymbol{\beta}\cdot\left[\left(\mathbf{s}+\frac{\gamma^2}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta}\right)\times\mathbf{B}\right]\boldsymbol{\beta} \\
&- \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{s})(\boldsymbol{\beta}\times\mathbf{B}) \\
&= \frac{e}{m}\frac{g}{2\gamma}\mathbf{s}\times\mathbf{B}+\frac{e}{m}(\boldsymbol{\beta}\cdot\mathbf{s})(\boldsymbol{\beta}\times\mathbf{B})\left[\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right] \\
&+ \frac{e}{m}\left[\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right]\boldsymbol{\beta}\cdot(\mathbf{s}\times\mathbf{B})\boldsymbol{\beta} \qquad (38)
\end{aligned}$$

is obtained. Finally, using $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\mathbf{d} = (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\times\mathbf{c})+(\mathbf{b}\cdot\mathbf{d})(\mathbf{c}\times\mathbf{a})+(\mathbf{c}\cdot\mathbf{d})(\mathbf{a}\times\mathbf{b})$, we can further simplify the above as

$$
\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM, }\mathbf{B}\text{ terms only}} &= \frac{e}{m}\frac{g}{2\gamma}\mathbf{s}\times\mathbf{B} + \frac{e}{m}(\boldsymbol{\beta}\cdot\mathbf{s})(\boldsymbol{\beta}\times\mathbf{B})\left[\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right] \\
&+ \frac{e}{m}\left[\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right]\big(\beta^2\mathbf{s}\times\mathbf{B} + (\boldsymbol{\beta}\cdot\mathbf{s})(\mathbf{B}\times\boldsymbol{\beta}) + (\mathbf{B}\cdot\boldsymbol{\beta})(\boldsymbol{\beta}\times\mathbf{S})\big) \\
&= \frac{e}{m}\left[\frac{g}{2\gamma} + \frac{g-2}{2}\frac{\gamma\beta^2}{\gamma+1}\right]\mathbf{s}\times\mathbf{B} + \frac{e}{m}\left[\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right](\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\times\mathbf{s}) \\
&= \frac{e}{m}\left[\frac{g-2}{2} + \frac{1}{\gamma}\right]\mathbf{s}\times\mathbf{B} + \frac{e}{m}\left[\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right](\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\times\mathbf{s}).
\end{aligned}
$$

Therefore, the MDM term becomes

$$
\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{MDM}} &= \frac{e}{m}\left[\left(\frac{g-2}{2} + \frac{1}{\gamma}\right)\mathbf{s}\times\mathbf{B} + \left(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\times\mathbf{s})\right. \\
&+ \left.\left(\frac{g-2}{2} + \frac{1}{\gamma+1}\right)\mathbf{s}\times\left(\frac{\mathbf{E}}{c}\times\boldsymbol{\beta}\right)\right]. \qquad (39)
\end{aligned}
$$

Note that there is a typo of sign in Eq. (18) of `arXiv:1308.1580v3`. Now, the EDM term becomes

$$
\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{EDM}} &= \left[\frac{d\mathbf{S}}{dt}\right]_{\text{EDM}} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\left[\frac{d\mathbf{S}}{dt}\right]_{\text{EDM}}\right)\boldsymbol{\beta} \\
&= -\frac{2cd}{\gamma}\left[(\mathbf{S}\cdot\boldsymbol{\beta})\mathbf{B} - \mathbf{S}\times\frac{\mathbf{E}}{c} + \frac{\boldsymbol{\beta}}{c}\gamma^2\Big\{-c\mathbf{S}\cdot\mathbf{B} - \boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{E}) + c(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})\Big\}\right] \\
&+ \frac{2cd}{\gamma+1}\left[(\mathbf{S}\cdot\boldsymbol{\beta})(\mathbf{B}\cdot\boldsymbol{\beta}) - \left(\mathbf{S}\times\frac{\mathbf{E}}{c}\right)\cdot\boldsymbol{\beta}\right. \\
&+ \left.\frac{\beta^2}{c}\gamma^2\Big\{-c\mathbf{S}\cdot\mathbf{B} - \boldsymbol{\beta}\cdot(\mathbf{S}\times\mathbf{E}) + c(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})\Big\}\right]\boldsymbol{\beta}. \qquad (40)
\end{aligned}
$$

Again, collecting terms with magnetic-field-only gives us

$$\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{EDM, }\mathbf{B}\text{ terms only}} &= 2cd\bigg[-\frac{1}{\gamma}(\mathbf{S}\cdot\boldsymbol{\beta})\mathbf{B}+(\mathbf{S}\cdot\mathbf{B})\boldsymbol{\beta}\Big(\gamma-\frac{\gamma^2\beta^2}{\gamma+1}\Big) \\
&+ (\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta}\Big(-\gamma+\frac{1}{\gamma+1}+\frac{\gamma^2\beta^2}{\gamma+1}\Big)\bigg] \\
&= 2cd\bigg[-\frac{1}{\gamma}(\mathbf{S}\cdot\boldsymbol{\beta})\mathbf{B}+(\mathbf{S}\cdot\mathbf{B})\boldsymbol{\beta}-\frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\mathbf{S})\boldsymbol{\beta}\bigg] \\
&= 2cd\bigg[-\frac{1}{\gamma}(\gamma\mathbf{s}\cdot\boldsymbol{\beta})\mathbf{B}+\Big(\mathbf{s}+\frac{\gamma^2}{\gamma+1}(\mathbf{s}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\Big)\cdot\mathbf{B}\boldsymbol{\beta} \\
&- \frac{\gamma}{\gamma+1}(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\cdot\gamma\mathbf{s})\boldsymbol{\beta}\bigg] \\
&= (-2cd)\mathbf{s}\times(\mathbf{B}\times\boldsymbol{\beta}).
\end{aligned} \tag{41}$$

There are in total four terms with the electric field, so

$$\begin{aligned}
\left[\frac{d\mathbf{s}}{dt}\right]_{\text{EDM, }\mathbf{E}\text{ terms only}} &= 2cd\Bigg[\frac{1}{\gamma}\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)+\gamma\boldsymbol{\beta}\cdot\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)\boldsymbol{\beta}-\frac{1}{\gamma+1}\boldsymbol{\beta}\cdot\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)\boldsymbol{\beta} \\
&- \frac{\gamma^2\beta^2}{\gamma+1}\boldsymbol{\beta}\cdot\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)\boldsymbol{\beta}\Bigg] \\
&= 2cd\Bigg[\frac{1}{\gamma}\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)+\frac{\gamma}{\gamma+1}\boldsymbol{\beta}\cdot\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)\boldsymbol{\beta}\Bigg] \\
&= 2cd\Bigg[\frac{1}{\gamma}\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big) \\
&+ \frac{\gamma}{\gamma+1}\bigg(\beta^2\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)+(\mathbf{S}\cdot\boldsymbol{\beta})\Big(\frac{\mathbf{E}}{c}\times\boldsymbol{\beta}\Big)+\Big(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\beta}\Big)(\boldsymbol{\beta}\times\mathbf{S})\bigg)\Bigg] \\
&= 2cd\Bigg[\Big(\frac{1}{\gamma}+\frac{\gamma\beta^2}{\gamma+1}\Big)\Big(\mathbf{S}\times\frac{\mathbf{E}}{c}\Big)+\frac{\gamma}{\gamma+1}(\mathbf{S}\cdot\boldsymbol{\beta})\Big(\frac{\mathbf{E}}{c}\times\boldsymbol{\beta}\Big) \\
&+ \frac{\gamma}{\gamma+1}\Big(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\beta}\Big)(\boldsymbol{\beta}\times\mathbf{S})\Bigg] \\
&= 2cd\Bigg[\Big(\mathbf{s}+\frac{\gamma^2}{\gamma+1}(\mathbf{s}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\Big)\times\frac{\mathbf{E}}{c}+\frac{\gamma^2}{\gamma+1}(\mathbf{s}\cdot\boldsymbol{\beta})\Big(\frac{\mathbf{E}}{c}\times\boldsymbol{\beta}\Big) \\
&+ \frac{\gamma}{\gamma+1}\Big(\frac{\mathbf{E}}{c}\cdot\boldsymbol{\beta}\Big)\boldsymbol{\beta}\times\Big(\mathbf{s}+\frac{\gamma^2}{\gamma+1}(\mathbf{s}\cdot\boldsymbol{\beta})\boldsymbol{\beta}\Big)\Bigg] \\
&= 2cd\Bigg[\mathbf{s}\times\frac{\mathbf{E}}{c}+\frac{\gamma}{\gamma+1}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)(\boldsymbol{\beta}\times\mathbf{s})\Bigg] \qquad (42)
\end{aligned}$$

is satisfied. Simplifying the above gives us

$$\left[\frac{d\mathbf{s}}{dt}\right]_{\text{EDM}} = 2cd\Bigg[\mathbf{s}\times\frac{\mathbf{E}}{c}+\frac{\gamma}{\gamma+1}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)(\boldsymbol{\beta}\times\mathbf{s})-\mathbf{s}\times(\mathbf{B}\times\boldsymbol{\beta})\Bigg]. \qquad (43)$$

If we summarize what we did so far, it becomes

$$\begin{aligned}\left[\frac{d\mathbf{s}}{dt}\right]_{\mathrm{MDM}} &= \frac{e}{m}\left[\left(\frac{g-2}{2}+\frac{1}{\gamma}\right)\mathbf{s}\times\mathbf{B}+\left(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})(\boldsymbol{\beta}\times\mathbf{s})\right.\\ &+ \left.\left(\frac{g-2}{2}+\frac{1}{\gamma+1}\right)\mathbf{s}\times\left(\frac{\mathbf{E}}{c}\times\boldsymbol{\beta}\right)\right],\\ \left[\frac{d\mathbf{s}}{dt}\right]_{\mathrm{EDM}} &= 2cd\left[\mathbf{s}\times\frac{\mathbf{E}}{c}+\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)(\boldsymbol{\beta}\times\mathbf{s})-\mathbf{s}\times(\mathbf{B}\times\boldsymbol{\beta})\right].\end{aligned} \tag{44}$$

If we change the constant term d in the EDM part as $d = e\eta/(4mc)$, [2] then $2cd = e\eta/2m$. The spin equation of motion is expressed as

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\omega}_s \times \mathbf{s} \tag{45}$$

where $\boldsymbol{\omega}_s$ is the angular velocity of the spin precession. So in our case,

$$\begin{aligned}\boldsymbol{\omega}_s &= -\frac{e}{m}\left[\left(\frac{g-2}{2}+\frac{1}{\gamma}\right)\mathbf{B}-\left(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta}-\left(\frac{g-2}{2}+\frac{1}{\gamma+1}\right)\left(\boldsymbol{\beta}\times\frac{\mathbf{E}}{c}\right)\right.\\ &+ \left.\frac{\eta}{2}\left(\frac{\mathbf{E}}{c}-\frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\boldsymbol{\beta}+\boldsymbol{\beta}\times\mathbf{B}\right)\right]\end{aligned} \tag{46}$$

is satisfied. We would like to point out again that the spin vector $\mathbf{s}$ is defined at the particle rest frame and $\mathbf{E}$ and $\mathbf{B}$ are defined in the laboratory frame.

The last step is to calculate the spin precession angular velocity with respect to the particle momentum direction. To do that, one has to calculate the angular velocity of the particle and subtract it from the spin precession angular velocity vector. For that, let's define $\mathbf{N} \equiv \boldsymbol{\beta}/\beta = \mathbf{p}/p$ and differentiate it with respect to time. Then, we get

$$\frac{d\mathbf{N}}{dt} = \frac{\dot{\boldsymbol{\beta}}}{\beta} - \frac{\boldsymbol{\beta}}{\beta^3}\beta\dot{\beta} = \frac{\dot{\boldsymbol{\beta}}}{\beta} - \frac{\boldsymbol{\beta}}{\beta^3}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) = \boldsymbol{\omega}_p \times \mathbf{N} \tag{47}$$

[2] Note that formally $d = \eta\frac{e}{2mc}\mathbf{s}$ where $\mathbf{s}$ is spin of muon. And for muon, $\mathbf{s} = \frac{\hbar}{2}$. So, $d = \eta\frac{e}{2mc^2}\frac{c\hbar}{2} = \eta\frac{3\times10^{10}(\mathrm{cm/s})\cdot \mathrm{e}\times 6.5\times10^{-22}\ \mathrm{MeV\ s}}{4\times207\times0.511\ \mathrm{MeV}} \approx \eta\times4.7\times10^{-14}$ e cm.

in general. Now, if we apply it to the particle in the electromagnetic field, we get

$$\begin{aligned}
\frac{d\mathbf{N}}{dt} &= \frac{\dot{\boldsymbol{\beta}}}{\beta} - \frac{\boldsymbol{\beta}}{\beta^3}(\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}) \\
&= \frac{e}{\beta\gamma m}\left[\boldsymbol{\beta}\times\mathbf{B} + \frac{\mathbf{E}}{c} - \boldsymbol{\beta}\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)\right] - \frac{e\boldsymbol{\beta}}{\beta^3\gamma m}\left[\boldsymbol{\beta}\cdot(\boldsymbol{\beta}\times\mathbf{B}) + \boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c} - \beta^2\Big(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\Big)\right] \\
&= \frac{e}{\gamma m}\left[-\mathbf{B}\times\mathbf{N} + \frac{1}{\beta}\frac{\mathbf{E}}{c} - \beta\mathbf{N}\Big(\mathbf{N}\cdot\frac{\mathbf{E}}{c}\Big) - \frac{1}{\beta}\mathbf{N}\Big(\mathbf{N}\cdot\frac{\mathbf{E}}{c}\Big) + \beta\mathbf{N}\Big(\mathbf{N}\cdot\frac{\mathbf{E}}{c}\Big)\right] \\
&= \frac{e}{\gamma m}\left[-\mathbf{B}\times\mathbf{N} + \frac{1}{\beta}\Big\{\frac{\mathbf{E}}{c} - \mathbf{N}\Big(\mathbf{N}\cdot\frac{\mathbf{E}}{c}\Big)\Big\}\right] \\
&= \frac{e}{\gamma m}\left[-\mathbf{B} + \frac{1}{\beta}\Big(\mathbf{N}\times\frac{\mathbf{E}}{c}\Big)\right]\times\mathbf{N}. \qquad (48)
\end{aligned}$$

So,

$$\boldsymbol{\omega}_p = \frac{e}{m\gamma}\left[\frac{1}{\beta}\Big(\mathbf{N}\times\frac{\mathbf{E}}{c}\Big) - \mathbf{B}\right] \qquad (49)$$

is obtained. Therefore the spin precession angular velocity with respect to the particle motion

$$
\begin{aligned}
\boldsymbol{\omega}_a &= \boldsymbol{\omega}_s - \boldsymbol{\omega}_p \\
&= -\frac{e}{m}\left[\left(\frac{g-2}{2}+\frac{1}{\gamma}\right)\mathbf{B} - \left(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(\frac{g-2}{2}+\frac{1}{\gamma+1}\right)\left(\boldsymbol{\beta}\times\frac{\mathbf{E}}{c}\right)\right. \\
&+ \left.\frac{\eta}{2}\left(\frac{\mathbf{E}}{c} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\boldsymbol{\beta} + \boldsymbol{\beta}\times\mathbf{B}\right)\right] \\
&- \frac{e}{m\gamma}\left(\frac{1}{\beta}\mathbf{N}\times\frac{\mathbf{E}}{c} - \mathbf{B}\right) \\
&= -\frac{e}{m}\left[\left(\frac{g-2}{2}\right)\mathbf{B} - \left(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(\frac{g-2}{2}+\frac{1}{\gamma+1}-\frac{1}{\beta^2\gamma}\right)\left(\boldsymbol{\beta}\times\frac{\mathbf{E}}{c}\right)\right. \\
&+ \left.\frac{\eta}{2}\left(\frac{\mathbf{E}}{c} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\boldsymbol{\beta} + \boldsymbol{\beta}\times\mathbf{B}\right)\right] \\
&= -\frac{e}{m}\left[\left(\frac{g-2}{2}\right)\mathbf{B} - \left(\frac{g-2}{2}\frac{\gamma}{\gamma+1}\right)(\boldsymbol{\beta}\cdot\mathbf{B})\boldsymbol{\beta} - \left(\frac{g-2}{2}-\frac{1}{\gamma^2-1}\right)\left(\boldsymbol{\beta}\times\frac{\mathbf{E}}{c}\right)\right. \\
&+ \left.\frac{\eta}{2}\left(\frac{\mathbf{E}}{c} - \frac{\gamma}{\gamma+1}\left(\boldsymbol{\beta}\cdot\frac{\mathbf{E}}{c}\right)\boldsymbol{\beta} + \boldsymbol{\beta}\times\mathbf{B}\right)\right] \qquad (50)
\end{aligned}
$$

is satisfied. The usual experimental conditions require

$$\mathbf{B}\cdot\boldsymbol{\beta} = 0, \quad \mathbf{E}\cdot\boldsymbol{\beta} = 0 \qquad (51)$$

and then

$$\boldsymbol{\omega}_a = -\frac{e}{m}\left[\left(\frac{g-2}{2}\right)\mathbf{B} - \left(\frac{g-2}{2}-\frac{1}{\gamma^2-1}\right)\left(\boldsymbol{\beta}\times\frac{\mathbf{E}}{c}\right) + \frac{\eta}{2}\left(\frac{\mathbf{E}}{c}+\boldsymbol{\beta}\times\mathbf{B}\right)\right] \qquad (52)$$

is finally obtained. This equation is commonly used in g-2/EDM experiments. For the muon, one defines $a_\mu \equiv (g-2)/2$ and then

$$\boldsymbol{\omega}_a = -\frac{e}{m}\left[a_\mu\mathbf{B} - \left(a_\mu - \frac{1}{\gamma^2-1}\right)\left(\boldsymbol{\beta}\times\frac{\mathbf{E}}{c}\right) + \frac{\eta}{2}\left(\frac{\mathbf{E}}{c}+\boldsymbol{\beta}\times\mathbf{B}\right)\right]. \qquad (53)$$

This must be the master formula used in the muon g-2/EDM experiments.

2 Interpretation of the Results

2.1 When EDM is Zero

When there is only $\mathbf{B}$ field that satisfies $\boldsymbol{\beta} \cdot \mathbf{B} = 0$ with no EDM, $\boldsymbol{\omega_s}$ becomes

$$\boldsymbol{\omega}_s = -\frac{e}{m}\left(a_\mu + \frac{1}{\gamma}\right)\mathbf{B} \tag{54}$$

according to our calculation. The first term is the muon spin precession in its rest frame and the second one is Thomas's precession because we have an relativistically accelerating frame. In this case, $\boldsymbol{\omega}_p$ becomes

$$\boldsymbol{\omega}_p = -\frac{e}{m}\frac{1}{\gamma}\mathbf{B} \tag{55}$$

. Therefore,

$$\boldsymbol{\omega}_a = \boldsymbol{\omega}_s - \boldsymbol{\omega}_p = -\frac{e}{m}a_\mu\mathbf{B} \tag{56}$$

where the relativistic effect is canceled out. If we have an electric field $\mathbf{E}$ that satisfies $\boldsymbol{\beta} \cdot \mathbf{E} = 0$, we get

$$\boldsymbol{\omega}_a = -\frac{e}{m}\left[a_\mu\mathbf{B} - \left(a_\mu - \frac{1}{\gamma^2 - 1}\right)\left(\boldsymbol{\beta} \times \frac{\mathbf{E}}{c}\right)\right]. \tag{57}$$

The reason of having terms with $\mathbf{E}$ field is due to the fact that the particle now sees the $\mathbf{B}$ field from moving $\mathbf{E}$ field with respect to the particle (known as the motional $\mathbf{B}$ field and is a relativistic effect).

2.2 When MDM is Zero

For the EDM term when no $\mathbf{B}$ field is presented (and $\boldsymbol{\beta} \cdot \mathbf{E} = 0$), we get

$$\boldsymbol{\omega}_a = -\frac{e}{m}\frac{\eta}{2}\frac{\mathbf{E}}{c} \tag{58}$$

and that is expected. Now if there is $\mathbf{B}$ field with $\boldsymbol{\beta} \cdot \mathbf{B} = 0$, one gets

$$\boldsymbol{\omega}_a = -\frac{e}{m}\frac{\eta}{2}\left[\frac{\mathbf{E}}{c} + \boldsymbol{\beta} \times \mathbf{B}\right]. \tag{59}$$

Again, we observe a motional $\mathbf{E}$ field due to the motion of the particle. You may ask why the motional fields appear differently in MDM and EDM? A short, perhaps an irresponsible answer is: we have shown why it is by an explicit calculation. But another, may be better answer is due to the fact that there is different treatment of $\mathbf{E}$ and $\mathbf{B}$ fields embedded in the Maxwell equations. Do you have better answer?