Derivation of the Inverse Radon Transformation

(Revised on Nov. 04, 2007 by E. Won)

I. THE RADON TRANSFORMATION

As far as our experimental approach is concerned, the Radon transformation is defined as

$$pr(q,\theta) = \int_{-\infty}^{+\infty} W(q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)\ dp.$$

Please note that our reference by S. Shinohara had typo ($p\sin\theta +$ $p\cos\theta$ should have been $q\sin\theta +$ $p\cos\theta$) and I corrected accordingly. First, we would like to transform the above to the other form. Let us put

$$x = q\cos\theta - p\sin\theta$$

then $dp = dx/(-\sin\theta)$ and

$$\begin{aligned} q\sin\theta + p\cos\theta &= q\frac{\sin^2\theta}{\sin\theta} + p\cos\theta \\ &= q\frac{1-\cos^2\theta}{\sin\theta} + p\cos\theta \\ &= \frac{q}{\sin\theta} - x\cot\theta. \end{aligned}$$

Therefore,

$$\begin{aligned} pr(q,\theta) &= \frac{1}{|\sin\theta|}\int_{-\infty}^{+\infty} W(x, \frac{q}{\sin\theta} - x\cot\theta)\ dx \\ &= \frac{1}{|\sin\theta|}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(x,y)\delta(y - \frac{q}{\sin\theta} + x\cot\theta)\ dxdy \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(x,y)\delta(-q + y\sin\theta + x\cos\theta)\ dxdy \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(x,y)\delta(q - x\cos\theta - y\sin\theta)\ dxdy \end{aligned}$$

where we used $1/|a|\delta(x) = \delta(ax)$. Since two expressions for $pr(q,\theta)$ are equal to each other, we conclude that

$$\begin{aligned} pr(q,\theta) &= \int_{-\infty}^{+\infty} W(q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)\ dp \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(x,y)\delta(q - x\cos\theta - y\sin\theta)\ dxdy. \end{aligned}$$

II. THE INVERSE RADON TRANSFORMATION

In our case, our inverse Radon transformation is achieved using the technique called a filtered backprojection, which is a mixture of the Fourier transformation and a coordinate transformation (polar coordinates). Let us proceed.

Let us define the two dimensional Fourier transformation of the Wigner function as

$$\widetilde{W}(k_x, k_y) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(q,p) \exp^{-i(k_x q + k_y p)} \; dq \; dp$$

so

$$W(q,p) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \widetilde{W}(k_x, k_y) \exp^{i(k_x q + k_y p)} \; dk_x \; dk_y$$

is satisfied as the inverse Fourier transformation. Now, with the following coordinate transformation

$$k_x = \xi \cos\theta$$
$$k_y = \xi \sin\theta$$

we get

$$\begin{aligned} W(q,p) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \widetilde{W}(k_x, k_y) \exp^{i(k_x q + k_y p)} \; dk_x \; dk_y \\ &= \int_{0}^{\pi} \int_{-\infty}^{+\infty} \widetilde{W}(\xi\cos\theta, \xi\sin\theta) \exp^{i\xi(q\cos\theta + p\sin\theta)} \; |\xi| d\xi \; d\theta. \end{aligned}$$

Now, let us get a form for $\widetilde{W}(\xi\cos\theta, \xi\sin\theta)$:

$$\begin{aligned} \widetilde{W}(\xi\cos\theta, \xi\sin\theta) &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(q,p) \exp^{-i\xi(q\cos\theta + p\sin\theta)} \; dq \; dp \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} W(q,p) \exp^{-i\xi x} \delta(x - q\cos\theta - p\sin\theta) \; dq \; dp \; dx \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \exp^{-i\xi x} pr(x,\theta) \; dx \end{aligned}$$

where at the last step, the result from the previous section was used. Now we are almost done. By putting the above equation into the inverse Fourier transformation, we get

$$\begin{aligned} W(q,p) &= \int_{0}^{\pi} \int_{-\infty}^{+\infty} \widetilde{W}(\xi\cos\theta, \xi\sin\theta) \exp^{i\xi(q\cos\theta + p\sin\theta)} \; |\xi| d\xi \; d\theta \\ &= \int_{0}^{\pi} \int_{-\infty}^{+\infty} \left[\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \exp^{-i\xi x} pr(x,\theta) \; dx\right] \exp^{i\xi(q\cos\theta + p\sin\theta)} \; |\xi| d\xi \; d\theta \\ &= \frac{1}{4\pi^2} \int_{0}^{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} pr(x,\theta) |\xi| \exp^{i\xi(q\cos\theta + p\sin\theta - x)} \; dx \; d\xi \; d\theta \end{aligned}$$

and is identical to the definition by S. Shinohara which is

$$
\begin{aligned}
W(q,p) &= \frac{1}{2\pi^2}\int_0^{\pi}\int_{-\infty}^{+\infty} pr(x,\theta)K(q\cos\theta + p\sin\theta - x)\ dx\ d\theta \\
K(y) &= \frac{1}{2}\int_{-\infty}^{+\infty} |\xi| \exp(i\xi y)\ dy.
\end{aligned}
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