# Reflections and Impedance Matching in a Transmission Line

**(Revised on Feb. 27, 2007 by E. Won)**

## I. TWO CONDUCTORS SEPARATED BY A DIELECTRIC AS A COAXIAL LINE

One can assume that two conductors separated by a dielectric as a simplified model for coaxial lines. In such configuration, coaxial cables contain a certain capacitance between signal and ground due to the dielectric. It also has self-inductance as we deal with time-varying current on the conductor. let us assme that $C_0$ and $L_0$ represent the capacitance and the self-inductance *per unit length* of the cable. From the first year electromagnetic theory, it is easy to show that

$$C_0 = \frac{2\pi\epsilon}{\ln(b/a)}[\mathrm{F/m}] \tag{1}$$

$$L_0 = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)[\mathrm{H/m}]. \tag{2}$$

## II. WAVE EQUATION FOR A COAXIAL LINE

Suppose we have a coaxial line with the resistance per unit length $R_0$, the inductance per unit length $L_0$, the capacitance per unit length $C_0$, and the conductance per unit length. Note that $R_0$ and $L_0$ are to be connected along the signal line, and $C_0$ and $G_0$ are to be connected between signal and background lines. We consider the voltage $V(x,t)$ and the current $I(x,t)$ in the cable. For an infinitesimal unit length of cable $dx$, let us consider $dV$ across this infinitesimal distance. There are two sources: one is due to the voltage drop $-R_0 dx\ I(x,t)$ due to the resistance $R_0$ and the other is the self-induced emf $-L_0\partial I/\partial t$ due to the time-varying current. Therefore, we can write them as

$$dV(x,t) = -R_0 I(x,t)\ dx - L_0\ dx\ \frac{\partial I(x,t)}{\partial t}$$

For the change of the current, there are also two sources. First one is from the Ohm's law, $-G_0\ dx\ V(x,t)$. Second one is the displacement current ($i_d$) through the capacitor $C_0$. From $i_d = (\epsilon_0 A)dE/dt$ and $C_0 = \epsilon_0 A/d$, we get $i_d = (C_0 d)dE/dt = C_0 dV/dt$. Therefore, it can be summarize as

$$dI(x,t) = -G_0 V(x,t)\ dx - C_0\ dx\ \frac{\partial V(x,t)}{\partial t}.$$

Both equations above now can be turned into the following two differential equations,

$$\frac{\partial V}{\partial x} = R_0 I - L_0 \frac{\partial I}{\partial t} \tag{3}$$

$$\frac{\partial I}{\partial x} = G_0 V - C_0 \frac{\partial V}{\partial t} \tag{4}$$

which are two coupled differential equations. By differentiating $V$ with respect to $x$ one more time, we get

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= -R_0 \frac{\partial I}{\partial x} - L_0 \frac{\partial^2 I}{\partial x \partial t} \\ &= -R_0 \left( -G_0 V - C_0 \frac{\partial V}{\partial t} \right) - L_0 \frac{\partial^2 I}{\partial x \partial t} \\ &= R_0 G_0 V + R_0 C_0 \frac{\partial V}{\partial t} - L_0 \frac{\partial^2 I}{\partial x \partial t} \end{aligned} \tag{5}$$

By differentiating Eq. (4) with respect to $t$ and having it to Eq. (5) we get

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= R_0 G_0 V + R_0 C_0 \frac{\partial V}{\partial t} - L_0 \left( -G_0 \frac{\partial V}{\partial t} - C_0 \frac{\partial^2 V}{\partial t^2} \right) \\ &= R_0 G_0 V + (L_0 G_0 + R_0 C_0) \frac{\partial V}{\partial t} + L_0 C_0 \frac{\partial^2 V}{\partial t^2} \end{aligned} \tag{6}$$

and similar analysis shows also for the current term as

$$\frac{\partial^2 I}{\partial x^2} = R_0 G_0 I + (L_0 G_0 + R_0 C_0) \frac{\partial I}{\partial t} + LC \frac{\partial^2 I}{\partial t^2} \tag{7}$$

which is an identical equation as for $V(x,t)$, in Eq. (6). This is the general wave equation for $V(x,t)$ and $I(x,t)$ for a coaxial cable.

## III. THE IDEAL LOSSLESS CABLE

Let us assume we have an ideal lossless calbe where $R_0$ and $G_0$ are zero, which is a good approximation for relatively short lengths (a few m). Then we get

$$\frac{\partial^2 V}{\partial x^2} = L_0 C_0 \frac{\partial^2 V}{\partial t^2} \tag{8}$$

and this becomes a well recognized wave equation in one dimension. We do the usual separation-of-variables technique to separate the time and the space components. Let's try $V(x,t) = V(x)e^{i\omega t}$. Then we get

$$\frac{d^2 V}{dx^2} = -\omega^2 L_0 C_0 V(x) \equiv -k^2 V(x) \tag{9}$$

and the space solutions are then of the form

$$V(x) = V_+ e^{-kx} + V_- e^{kx} \tag{10}$$

and the overall solution becomes

$$V(x,t) = V_+ e^{i(\omega t - kx)} + V_- e^{i(\omega t + kx)}. \tag{11}$$

This represents two waves, one traveling in the $+x$ directionm ($V_+$), and the other in the $-x$ direction ($V_-$). If we assume the initial wave is running from left to right and that is a positive $x$ direction, then the second term corresponds to a reflection and its presence or absence depending on the boundary conditions for the particular situation we consider.

We can associate the velocity of the propagation of the signal to $L_0$ and $C_0$ as

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{L_0 C_0}}. \tag{12}$$

If we are interested in the time of propergation per unit length, we may get

$$T = \frac{1}{v} = \sqrt{L_0 C_0}. \tag{13}$$

If we use typical values of $L_0 = 1/4$ $\mu$H/m and $C_0 = 100$ pF/m, we get $T = 5$ ns/m for standard 50 $\Omega$ cables in our lab.

Now, let us calculate the impedance of the coaxial cable, known as *characteristic impedance*. It is the ratio of $V$ and $I$ of the cable, i.e.,

$$Z_0 = \frac{V}{I}.$$

For an ideal lossless cable with $V(x,t) = V_+ e^{i(kx - \omega t)}$, we have from Eq. (3),

$$\frac{\partial V}{\partial x} = -L_0 \frac{\partial I}{\partial t}$$

and it becomes

$$\begin{aligned}
\frac{\partial V}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial x} &= -L_0 \frac{\partial I}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial t} \\
&= L_0 v \frac{\partial I}{\partial (x - vt)} \\
&= L_0 \frac{1}{\sqrt{L_0 C_0}} \frac{\partial I}{\partial (x - vt)} \\
&= \sqrt{\frac{L_0}{C_0}} \frac{\partial I}{\partial (x - vt)}.
\end{aligned}$$

Integrating over $(x - vt)$, we get

$$Z_0 = \sqrt{\frac{L_0}{C_0}} \tag{14}$$

for an ideal lossless cable. It is the impedance offered to the propagation of the signal in the coaxial line. Now, if we recall Eq. (1) and Eq. (2), the characteristic impedance becomes

$$Z_0 = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\log\frac{b}{a}\ [\Omega] \tag{15}$$

where again $a$ and $b$ are the inner and outer radius of two conductors. If we write $\epsilon = \epsilon_r\epsilon_0$ where $\epsilon_r$ is the relative permittivity (dielectric constant) of a material and $\mu = \mu_r\mu_0$ $\mu_r$ is the relative permeability, the formula becomes

$$\begin{aligned} Z_0 &= \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}}\log\frac{b}{a} \\ &= \frac{1}{2\pi}\sqrt{\frac{\mu_r}{\epsilon_r}}\log\frac{b}{a}\sqrt{\frac{\mu_0}{\epsilon_0}} \end{aligned} \tag{16}$$

where

$$\sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6\ \Omega.$$

Typically, the ratio $b/a$ varies between 2 and $10^2$ and for a laboratory cable using polythene $Z_0 \sim 50\ \Omega$.

## IV. REFLECTIONS, CABLE TERMINATION, AND IMPEDANCE MATCHING

Suppose that a coaxial cable has the characteristic impedance $Z_0$ with a load of impedence $Z_L$. A wave traveling to the right $(V_+,\ I_+)$ may be reflected to produce a wave $(V_-,\ I_-)$. The boundary condition at $Z_L$ must be

$$\begin{aligned} V_+ + V_- &= V_L \\ I_+ + I_- &= I_L \\ \frac{V_+}{I_+} &= Z_0 \\ \frac{V_-}{I_-} &= -Z_0 \\ \frac{V_L}{I_L} &= -Z_L. \end{aligned}$$

Now, we are interested in $V_-/V_+$, the voltage ratio of reflected to incoming wave. It can be reduced as

$$Z_L = \frac{V_+ + V_-}{I_+ + I_-} = \frac{V_+ + V_-}{V_+ - V_-} Z_0 = \frac{1 + V_-/V_+}{1 - V_-/V_+} Z_0.$$

If we solve for $V_-/V_+$, we get

$$\begin{aligned} \frac{Z_L}{Z_0} &= \frac{1 + V_-/V_+}{1 - V_-/V_+} \\ \left(1 + \frac{V_-}{V_+}\right)\frac{Z_L}{Z_0} &= 1 + \frac{V_-}{V_+} \\ \frac{V_-}{V_+} &= \frac{Z_L - Z_0}{Z_L + Z_0}. \end{aligned}$$

Therefore, in order to have no reflection, $Z_L = Z_0$ has to be satisfied. So the characteristic impedance of 50 Ω lemo cable needs 50 Ω termination for the impedance matching and we have plenty of 50 Ω terminator in our lab.

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[1] H. J. Pain, The Physics of Vibrations and Waves, 6th ed., Wiley.