Ch04 Electric Fields in Matter

Matter $\begin{cases} \text{conductors} \\ \text{insulators (or dielectrics)} \end{cases}$

External electric field can induce dipole on neutral atoms.

$$\vec{p} = \alpha \vec{E}$$

dipole moment ($\vec{p}$), atomic polarizability (α), external electric field ($\vec{E}$).

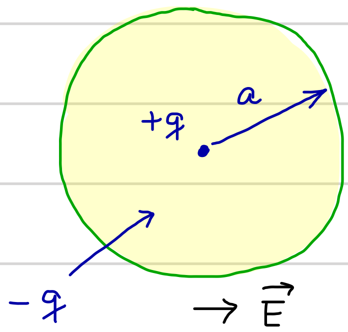
Unit of α?

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau' \rightarrow [p] = C \cdot m$$

so, $\vec{p} = \alpha \vec{E}$ gives us $C \cdot m = [\alpha] \frac{1}{4\pi\epsilon_0} \frac{C}{m^2}$ $\rightarrow$ $\therefore \frac{[\alpha]}{4\pi\epsilon_0} = m^3$

Ex) $\frac{\alpha(H)}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\ m^3$

Ex 4.1) A primitive model for an atom.



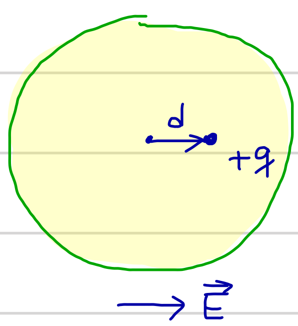
: consists of a point nucleus (+q) surrounded by a uniformly charged spherical cloud (−q) of radius a.

Calculate the atomic polarizability of such an atom.

⌐ The external field $\vec{E}$: +q (nucleus) shifted to right

−q (cloud) 〃 to left

→ Let's assume that equilibrium occurs +q is displaced a distance d from the center of the sphere.



The field at a distance d from the center is (q' : charge in radius d)

$$E_e = \frac{1}{4\pi\epsilon_0} \frac{q'}{d^2} = \frac{1}{4\pi\epsilon_0} q \cdot \frac{d^3}{a^3} = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

The external field (E) pushing +q to the right exactly balances the field produced by the electron cloud.

$$\therefore E = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}, \quad p = qd = (4\pi\epsilon_0 a^3) E$$

$$\therefore \alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v$$

(v: volume of the atom)

$$\frac{\alpha}{4\pi\epsilon_0} = a^3 = (10^{-10}\ m)^3 = 10^{-30}\ m^3$$

For real molecules, the atomic polarizability depends on the direction of $\vec{E}$ field.

So, most general linear relation between $\vec{E}$ and $\vec{P}$:

$$P_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

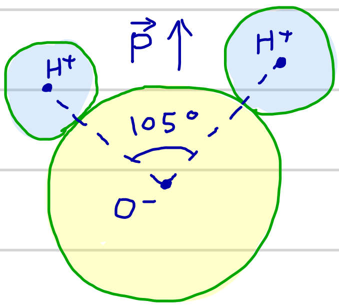
$$P_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

$$P_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$

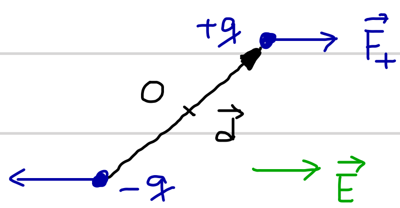
α_{ij} : polarizability tensor

4.1.3 ■ Alignment of polar Molecules

Let's discuss force due to external field on polar molecules.



Let's look at the force on net positive charge $(+q)$ and on net negative charge $(-q)$



Forces cancel exactly but there will be a torque $(\vec{N})$

$$\vec{N} = \vec{r}_+ \times \vec{F}_+ + (\vec{r}_- \times \vec{F}_-)$$

$$= \left[\left(\frac{\vec{d}}{2}\right) \times (q\vec{E})\right] + \left[\left(-\frac{\vec{d}}{2}\right) \times (-q\vec{E})\right] = q\,\vec{d} \times \vec{E} \quad \text{or}$$

$$\boxed{\vec{N} = \vec{p} \times \vec{E}} \qquad (\vec{p} = q\vec{d})$$

If the field is nonuniform, $(\vec{F}_+ + \vec{F}_- \neq 0)$

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q(\vec{E}_+ - \vec{E}_-) = q(\Delta\vec{E})$$

where $\Delta\vec{E}$: difference between field at $(+)$ and $(-)$

$\Delta E_x \equiv (\vec{\nabla} E_x) \cdot \vec{d}$ and same for E_x and E_y. Or

$$\Delta\vec{E} = (\vec{d} \cdot \vec{\nabla})\,\vec{E}$$

and therefore

$$\boxed{\vec{F} = (\vec{p} \cdot \vec{\nabla})\,\vec{E}}$$

4.1.4 Polarization

Let's define the polarization vector $\vec{P}$ ← capital letter

Polarization $\vec{P} \equiv$ dipole moment per unit volume.

Suppose we have polarized material : contains microscopic dipoles lined up.

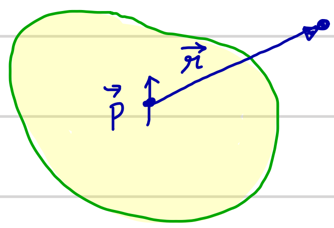
Q : What is the electric field produced by this object?

For a single dipole,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{\mathscr{r}}}{\mathscr{r}^2}$$

For continuous distribution, $\vec{P} \equiv \frac{\vec{p}}{d\tau'}$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(\vec{r}\,')\cdot\hat{\mathscr{r}}}{\mathscr{r}^2} d\tau'$$



So the introduction of polarization vector is now clear (distribution of dipoles)

Now, using $\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

(differentiation is w.r.t $\vec{r}\,'$) (Note: $\vec{\nabla}\left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$)

$$V = \frac{1}{4\pi\epsilon_0} \int_V \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\vec{\nabla}'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right) d\tau' - \int_V \frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})\, d\tau' \right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\underbrace{\oint_S \frac{1}{\mathscr{r}} \vec{P}\cdot d\vec{a}}_{\text{electric potential term with surface "charge"}} - \underbrace{\int_V \frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})\, d\tau'}_{\text{term with volume charge density.}} \right]$$

$\sigma_b \equiv \vec{P}\cdot\hat{n}$ (bound surface charge)

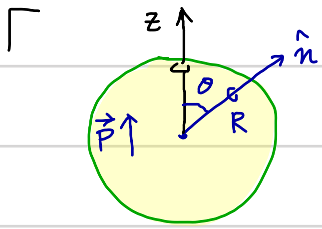
$\hat{n}$: normal unit vector

$\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$ (bound charge density)

So we get $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{\mathscr{r}} da' + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{\mathscr{r}} d\tau'$

→ A polarized object has electric potential from ρ_b and from σ_b.

Example 4.2) Find $\vec{E}$ produced by a uniformly polarized sphere of radius R.



$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$ since the sphere is uniformly charged.

$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$ ← This is same form as in E. 3.9.

($\sigma_0(\theta) = k\cos\theta$)

From example 3.9), we know that

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0} r\cos\theta & (r<R) \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta & (r\geq R) \end{cases}$$

Since $r\cos\theta = z$,

For $r<R$, $\vec{E} = -\vec{\nabla}V = -\dfrac{P}{3\epsilon_0}\hat{z} = -\dfrac{1}{3\epsilon_0}\vec{P}$.

For $r>R$,

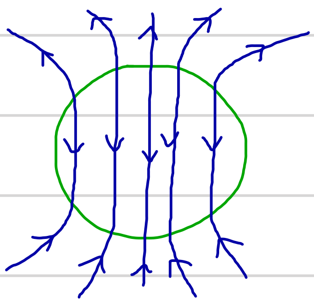
$$V(r,\theta) = \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta \qquad \text{with} \qquad \vec{p} = \frac{4}{3}\pi R^3 \vec{P}$$

dipole moment ($\vec{p}$) polarization vector ($\vec{P}$)

$$V = \frac{1}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta \frac{p}{\frac{4}{3}\pi R^3}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

and is identical to the potential for a perfect dipole at the origin.



σ_b ($= P\cos\theta$) builds up on surface.

$E^{\perp}$: discontinuous

4.3 ▪ The electric displacement

We will discuss Gauss's law in the presence of dielectrics.

First, polarization produces accumulation of bound charge.

$$\begin{cases} \rho_b = -\vec{\nabla}\cdot\vec{P} & \text{within the dielectric} \\ \sigma_b = \vec{P}\cdot\hat{n} & \text{on the surface} \end{cases}$$

We also introduce "free charge" ρ_f for the rest of charges (not result of polarization)

Within dielectrics, the total charge density can be

$$\rho = \rho_b + \rho_f$$

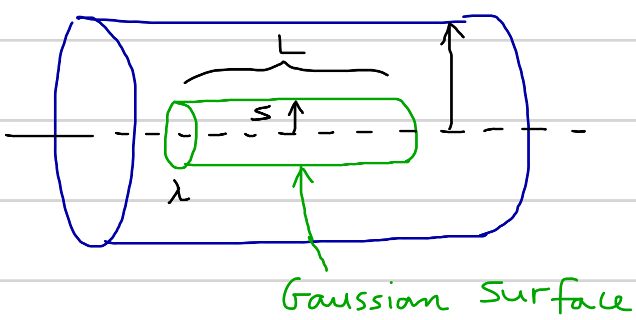
and Gauss's law reads

$$\epsilon_0 \underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{total field}} = \rho = \rho_b + \rho_f = -\vec{\nabla}\cdot\vec{P} + \rho_f$$

$\rightarrow \vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f$ and we define $\underbrace{\vec{D}}_{\text{electric displacement}} \equiv \epsilon_0\vec{E} + \vec{P}$

$\vec{\nabla}\cdot\vec{D} = \rho_f$ or $\oint_S \vec{D}\cdot d\vec{a} = Q_{fenc}$ $\quad Q_{fenc}$: total free charge enclosed in the volume.

Ex 4.4 A long straight wire with line charge density λ is surrounded by rubber insulation out to radius a. $\vec{D} = ?$



Applying Gauss's law

$$D(2\pi s L) = \lambda L$$

$$\therefore \vec{D} = \frac{\lambda}{2\pi s}\hat{s}$$

In the outside, $\vec{P} = 0$ so $\vec{E} = \frac{1}{\epsilon_0}\vec{D} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$ $\quad (s > a)$

Inside the rubber, $\vec{E}$ cannot be determined, since $\vec{P}$ is not given.

Two things about $\vec{D}$

- Even if $\vec{\nabla}\cdot\vec{D} = \rho_f$, $\quad \vec{D} \neq \frac{1}{4\pi}\int_V \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho_f(\vec{r}')d\tau'$

  because there is no Coulomb's law for $\vec{D}$.
- $\vec{\nabla}\times\vec{E} = 0$ but $\vec{\nabla}\times\vec{D} = \epsilon_0(\vec{\nabla}\times\vec{E}) + (\vec{\nabla}\times\vec{P}) = \vec{\nabla}\times\vec{P} \neq 0$

Boundary conditions for $\vec{D}$.

The previous case of boundary conditions :

$$\vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below}$$

$$\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{1}{\epsilon_0}\sigma$$

Now, for $\vec{D}$, from $\oint_S \vec{D}\cdot d\vec{a} = Q_{f_{enc}}$. So for the perpendicular to an interface

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

while $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$ gives

$$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$$

4.4 ▪ Linear Dielectrics

The polarization is usually proportional to $\vec{E}$:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \quad \text{total field}$$

χ_e : electric susceptibility

Materials obeying $\vec{P} = \epsilon_0\chi_e\vec{E}$ is called linear dielectrics.

Since $\vec{E}$ is the total field, one cannot compute $\vec{P}$ from $\vec{P} = \epsilon_0\chi_e\vec{E}$.

In linear media, we have

$$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1+\chi_e)\vec{E}$$

$\therefore$ $\vec{D} \propto \vec{E}$ and by defining $\epsilon \equiv \epsilon_0(1+\chi_e)$

$$\vec{D} = \epsilon\vec{E}$$

ϵ : permittivity of the material

$\epsilon_r \equiv 1+\chi_e = \dfrac{\epsilon}{\epsilon_0}$

$\rightarrow$ relative permittivity or dielectric constant.

4.4 Linear Dielectrics 187

Material	Dielectric Constant	Material	Dielectric Constant
Vacuum	1	Benzene	2.28
Helium	1.000065	Diamond	5.7-5.9
Neon	1.00013	Salt	5.9
Hydrogen (H_2)	1.000254	Silicon	11.7
Argon	1.000517	Methanol	33.0
Air (dry)	1.000536	Water	80.1
Nitrogen (N_2)	1.000548	Ice (-30° C)	104
Water vapor (100° C)	1.00589	$KTaNbO_3$ (0° C)	34,000

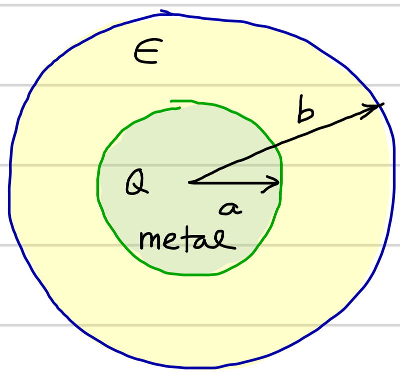
TABLE 4.2 Dielectric Constants (unless otherwise specified, values given are for 1 atm, 20° C). *Data from Handbook of Chemistry and Physics*, 91st ed. (Boca Raton: CRC Press, 2010).

Example 4.5. A metal sphere of radius a carries a charge Q (Fig. 4.20). It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution

Ex 4.5) A metal sphere of radius a carries a charge Q. It is surrounded by out to radius b, by a linear dielectric material of permittivity ϵ.

Potential at the center?



Starting with $\oint_S \vec{D}\cdot d\vec{a} = Q_{f\,enc}$,

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \quad \text{for } r>a$$

Since $\vec{D} = \epsilon\vec{E}$, we get

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{r} & \text{for } a<r<b \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & \text{for } r>b \end{cases}$$

The potential at the center

Note: $\vec{E} \propto \hat{r}$, $d\vec{l} = (-\hat{r})\,dl = (-\hat{r})(-dr) = \hat{r}\,dr$

$\therefore \vec{E}\cdot d\vec{l} = E\,dr$

$$V = -\int_\infty^0 \vec{E}\cdot d\vec{l} = -\int_\infty^b \left(\frac{Q}{4\pi\epsilon_0 r^2}\right)dr - \int_b^a \left(\frac{Q}{4\pi\epsilon}\frac{1}{r^2}\right)dr - \int_a^0 0\cdot dr$$

$$= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right) \text{ is obtained.}$$

The polarization $\vec{P} = \epsilon_0\chi_e\vec{E} = \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{r}$ in the dielectric, and

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$$

Note: $\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^{(3)}(\vec{r})$

and for $a<r<b$, $\vec{\nabla}\cdot\vec{P} = 0$.

While, $\sigma_b = \vec{P}\cdot\hat{n}$ gives

$$\sigma_b \begin{cases} = \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon b^2} & \text{outer surface} \\ = -\dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon a^2} & \text{inside surface} \end{cases}$$

4.4.2 Boundary Value Problems with Dielectrics.

In a linear dielectric, we have

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\epsilon_0\chi_e\vec{E}\right) \qquad \epsilon = \epsilon_0(1+\chi_e)$$

$$= -\vec{\nabla}\cdot\left(\epsilon_0\chi_e\frac{\vec{D}}{\epsilon}\right) = -\frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

If free charges are not imbeded in the material ($\rho_f = 0$), then $\rho_b = 0$.

Within such a dielectric, then, the potential obey Laplace's equation.

The formula on the boundary condition for $\vec{D}$:

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$\rightarrow \epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f$ or in terms of potential

$\epsilon_{above} \frac{\partial V_{above}}{\partial n} - \epsilon_{below} \frac{\partial V_{below}}{\partial n} = -\sigma_f$ whereas the potential itself is continuous $V_{above} = V_{below}$

Ex. 4.7) A sphere of homogeneous linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_0$. Find the electric field inside?

Let's list boundary conditions for $V(r,\theta)$ when $r \geq R$.

(i) $V_{in} = V_{out}$ at $r=R$

(ii) $\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r}$ at $r=R$ $(\sigma_f = 0)$

(iii) $V_{out} \rightarrow -E_0 r\cos\theta$ for $r \gg R$.

So, inside the sphere, we have

$$V_{in} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

and outside the sphere

$$V_{out} = -E_0 r\cos\theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

Now, the boundary condition at $r=R$ gives us

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = -E_0 R \underbrace{\cos\theta}_{= P_1(\cos\theta)} + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

When $l=1$: $A_1 R = -E_0 R + \frac{B_1}{R^2}$

" $l \neq 1$: $A_l R^l = \frac{B_l}{R^{l+1}}$

Using the boundary condition (ii), we get

$$\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta)$$

$$\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = -E_0 \cos\theta - \sum_{l=0}^{\infty} \frac{(l+1)}{R^{l+2}} B_l P_l(\cos\theta)$$

$\epsilon_r = \frac{\epsilon}{\epsilon_0}$

When $l=1$: $\epsilon_r A_1 = -E_0 - \frac{2}{R^3} B_1$

" $l \neq 1$: $\epsilon_r l A_l R^{l-1} = -\frac{(l+1)}{R^{l+2}} B_l$

For $l \neq 1$, from (i), $A_l R^l = \frac{B_l}{R^{l+1}} \rightarrow B_l = A_l R^{2l+1}$

$$\therefore \epsilon_r l A_l R^{l-1} = -\frac{(l+1) B_l}{R^{l+2}} = \frac{-(l+1)}{R^{l+2}} \cdot A_l R^{2l+1}$$

$$\rightarrow A_l \left[\epsilon_r l R^{l-1} + (l+1) R^{l-1} \right] = 0 \text{ for all } l \neq 1.$$

$\therefore A_l = 0$ so $B_l = 0$ for all $l \neq 1$.

For $l=1$:

$$\begin{cases} A_1 R = -E_0 R + \dfrac{B_1}{R^2} \\ \epsilon_r A_1 = -E_0 - \dfrac{2B_1}{R^3} \end{cases}$$

$$\rightarrow \quad A_1 R = -E_0 R + \frac{R}{2}\left(-\epsilon_r A_1 - E_0\right)$$

$$R\left(1+\frac{\epsilon_r}{2}\right) A_1 = -\frac{3}{2} E_0 R \quad \rightarrow \quad A_1 = -\frac{3}{2+\epsilon_r} E_0$$

$$B_1 = A_1 R^3 + E_0 R^3 = \frac{\epsilon_r - 1}{\epsilon_r + 2} E_0 R^3$$

So, $V_{in}(r,\theta) = A_1 r^1 P_1(\cos\theta) = -\dfrac{3}{2+\epsilon_r} E_0 \, r\cos\theta = -\dfrac{3E_0}{2+\epsilon_r} z$

$\vec{E}_{in} = \dfrac{3\vec{E}_0}{2+\epsilon_r}$ $\quad \therefore$ electric field inside is uniform.

In vacuum, $\epsilon \to \epsilon_0$, $\epsilon_r \to 1$ so $\vec{E}_{in} = \vec{E}_0$.

$0 \le \dfrac{3}{2+\epsilon_r} \le 1$ so $E_{in} \le E_0$ also.

4.4.3 • Energy in Dielectric Systems

It takes work to charge up a capacitor: $W = \frac{1}{2} C V^2$.

If the capacitor is filled with linear dielectric constant, we have

$C = \epsilon_r C_{vac}$ (In freshman physics, $K = \epsilon_r$ and the capacitance is increased.)

and the capacitance is increased by ϵ_r.

→ The work to charge up the capacitor is also increased.

($\because$ There is more free charges to store to achieve a given potential since $\vec{E}$ is canceled by bound charges)

In vacuum, the energy stored in any electrostatic system is

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau$$

→ How is it going to be changed in the presence of linear dielectrics?

From above, $W_0 = \frac{1}{2} C V^2 \xrightarrow{C \to \epsilon_r C_{vac}} W = \epsilon_r \frac{1}{2} C V^2$.

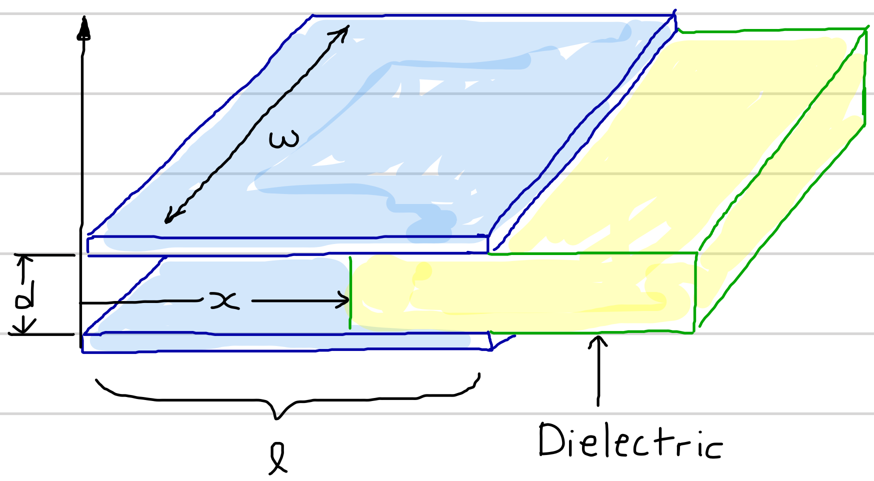
So, $W = \dfrac{\epsilon_0}{2}\int E^2 d\tau \rightarrow \dfrac{\epsilon_0}{2}\int \epsilon_r E^2 d\tau = \dfrac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$?

$$W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$$

4.4.4 Forces on Dielectrics

So far, we assumed that $\vec{E}$ field inside a parallel capacitor is uniform. In reality, it is not, and there is fringing field around edges.

Let's consider the following situation.



W : energy of the system above (may be a function of x)

If one pulls out it by dx, the energy change is equal to the work done:

$$dW = F_{me}\, dx \quad , \quad F_{me} = -F$$

electrostatic force on the dielectric.

$$\rightarrow \therefore F = -\frac{dW}{dx}$$

The energy stored in the capacitor is $W = \frac{1}{2}CV^2$

Now, what is the value of C in this case?

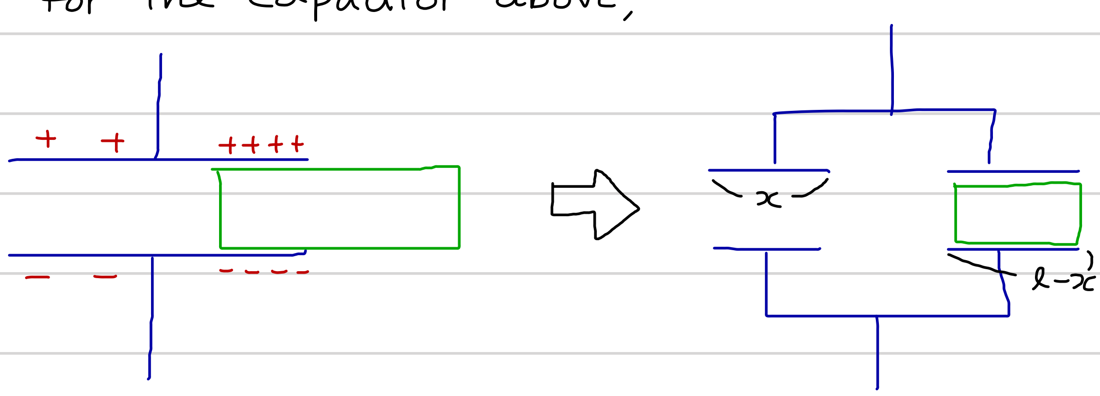
In vacuum, $\oint_S \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0} \rightarrow E \cdot A = \frac{Q}{\epsilon_0}$

V/d

$$\rightarrow Q = \frac{\epsilon_0 A}{d} V = CV$$

$$\therefore C = \frac{\epsilon_0 A}{d}$$

Now, for the capacitor above,



$\therefore\ C_{left} = \frac{\epsilon_0}{d} w x, \quad C_{right} = \epsilon_r \cdot \frac{\epsilon_0}{d} w(l-x)$

$C = C_{left} + C_{right} = \frac{\epsilon_0 w}{d}\left[x + \epsilon_r(l-x)\right]$

$= \frac{\epsilon_0 w}{d}\left[\epsilon_r l - (\epsilon_r - 1)x\right]$ $\quad \epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$

$= \frac{\epsilon_0 w}{d}\left[\epsilon_r l - \chi_e x\right]$ is obtained.

From $W = \frac{1}{2}\frac{Q^2}{C}$,

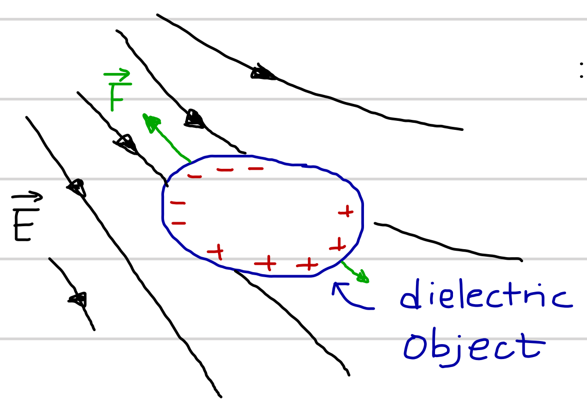
$F = -dW/dx = -\frac{1}{2}Q^2\left(-\frac{1}{C^2}\right)\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$

$\frac{dC}{dx} = \frac{\epsilon_0 w}{d}(-\chi_e)$

$\therefore\ F = -\frac{1}{2}V^2\frac{\epsilon_0 w}{d}\chi_e = -\frac{\epsilon_0 \chi_e w}{2d}V^2$

$(-)$: dielectric is pulled into the capacitor.

Note: a dielectric is always drawn from a region of weak field toward a region of stronger field.



: Net force is upward because the field is stronger upward.

(Feynman, Fig 10-8)