Electrodynamics

Ohm's law

$\vec{J}$: current density

$\vec{f}$: force per unit charge

$\vec{J} = \sigma \vec{f}$ $\quad$ σ : conductivity $\quad$ $\sigma = \infty$: perfect conductor

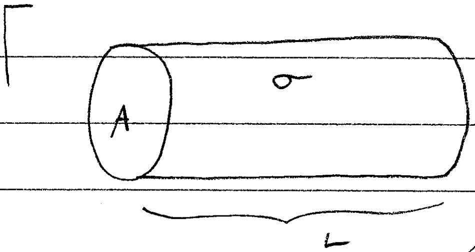
$= 0$: insulators

$\rho \equiv 1/\sigma$: resistivity

From Lorentz force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ $v \ll 1$, can be neglected.

$$\vec{f} = \vec{F}/q = \vec{E} = \frac{1}{\sigma}\vec{J} \longrightarrow \vec{J} = \sigma\vec{E} \text{ : Ohm's law}$$

Ex 7.1)



σ, A, L given. What is the current for given potential difference V?

$$I = JA = \sigma \underbrace{E}_{V/L} A = \frac{\sigma A}{L} V \qquad R = \frac{L}{\sigma A}$$

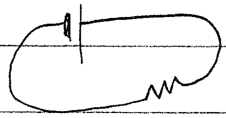
✓ Another form of Ohm's law : $V = IR$

✓ For steady currents and uniform conductivity,

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\sigma}\vec{\nabla}\cdot\vec{J} = 0$$

7.1.2 Electromotive force ← confusing

In a circuit w/ battery for example,



① battery provides force/charge ($\equiv \vec{f}_s$ ← source)

② There is an electrostatic force/q ($=\vec{E}$) (smooth out the flow)

So, the force/charge $\vec{f} = \vec{f}_s + \vec{E}$ $\quad$ $= 0$ ($\because \vec{\nabla}\times\vec{E} = 0$!)

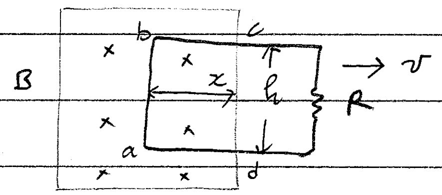
electromotive force $\mathcal{E} \equiv \oint \vec{f}\cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} + \oint \vec{E}\cdot d\vec{l}$

For ideal battery ($\vec{f} = \frac{1}{\sigma}\vec{J}$ w/ $\sigma \to \infty$, so $\vec{f} = 0$)

$$V = -\int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \vec{f}_s \cdot d\vec{l} = \oint \vec{f}_s \cdot d\vec{l} = \mathcal{E}$$

($\because$ $\vec{f}_s = 0$ outside source) confusing

Motional Emf



- charge segment ab experiences magnetic force

$$\vec{F} = q\,\vec{v} \times \vec{B}$$

$$\mathcal{E} = \oint \vec{f}_{mag} \cdot d\vec{\ell}$$

$$= \oint (\vec{v} \times \vec{B}) \cdot d\vec{\ell} = vBh$$

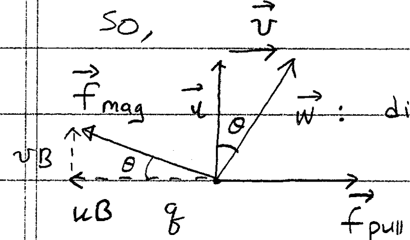
- $\vec{B}$ does no work
- Free charge moves to right w/ $\vec{v}$
- Due to $q\,\vec{u} \times \vec{B}$, free charge move vertically w/ $\vec{u}$

  → magnetic force to the left

  ↓ to counteract this, the person pulling the wire must exert a force/charge

$$f_{pull} = uB \quad \text{to the right}$$

So,



$\vec{w}$: direction of resultant velocity $(\vec{u} + \vec{v})$

- The work done/charge

$$\int \underbrace{\vec{f}_{pull}}_{uB} \cdot \underbrace{d\vec{\ell}}_{\frac{h}{\cos\theta}} = uB \frac{h}{\cos\theta} \sin\theta = u \tan\theta \, Bh = vBh = \mathcal{E}$$

$\sin\theta$

Let $\Phi \equiv \int \vec{B} \cdot d\vec{a}$ (flux of $\vec{B}$). In our case,

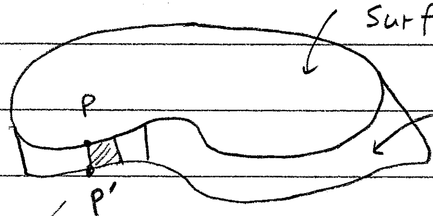
$\Phi = Bhx$, and

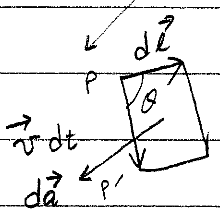
$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv$ (flux decreases)

So, $\mathcal{E} = -\frac{d\Phi}{dt}$ is obtained.

Now, $\mathcal{E} = -\frac{d\Phi}{dt}$ true for general case?

Loop of wire at time t, and at $t+dt$





The change in flux $d\Phi = \Phi(t+dt) - \Phi(t)$

$\cong \Phi_{\text{ribbon}}$ (when $dt \to 0$)

$$= \int_{\text{ribbon}} \vec{B} \cdot d\vec{a}$$

$\vec{v}$: velocity of the wire

$\vec{u}$: " charge in the wire

$\vec{w} = \vec{v} + \vec{u}$

면적분

$$d\vec{a} = (\vec{v} \times d\vec{l})\,dt \longrightarrow d\Phi = \int_{\text{ribbon}} \vec{B} \cdot d\vec{a}$$

$$\cong \oint \vec{B} \cdot (\vec{v} \times d\vec{l})\,dt$$

$$\rightarrow \frac{d\Phi}{dt} = \oint \vec{B} \cdot (\vec{v} \times d\vec{l})$$

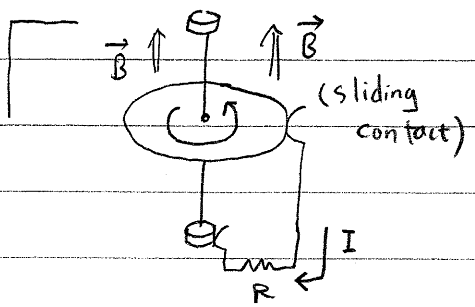
선적분, along $d\vec{l}$

$\vec{w} = \vec{u} + \vec{v}$ and $\vec{u} \parallel d\vec{l}$, so $\vec{v} \times d\vec{l} = \vec{w} \times d\vec{l}$

$$\therefore \frac{d\Phi}{dt} = \oint \vec{B} \cdot (\vec{w} \times d\vec{l}) = -\oint (\vec{w} \times \vec{B}) \cdot d\vec{l}$$

$$= -\oint \vec{f}_{mag} \cdot d\vec{l} = -\mathcal{E}$$

Ex 7.4 A metal disk of radius a, rotates w/ angular velocity ω, through $\vec{B}$ (uniform). I inside R?



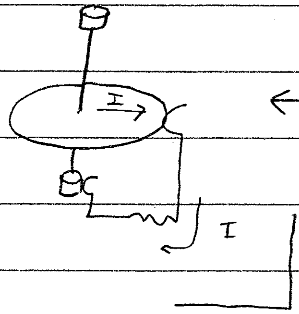
The speed of a point at s : $v = \omega s$

$\therefore \vec{f}_{mag} = \vec{v} \times \vec{B} = \omega s B \hat{s}$.

The emf is $\mathcal{E} = \int_0^a f_{mag}\, ds = \omega B \int_0^a ds$

$= \frac{\omega B}{2} a^2$

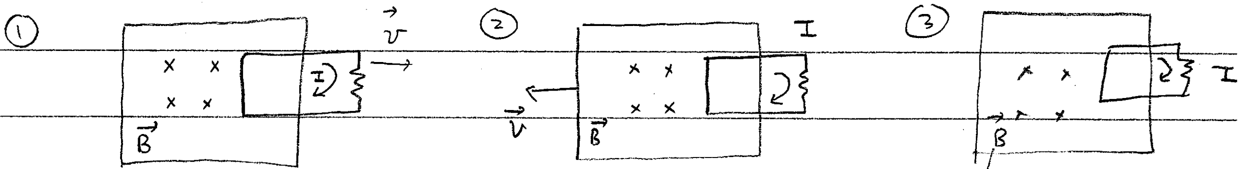
$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$



← "Faraday disk"

7.2 Electromagnetic Induction

· Faraday's Law



$\mathcal{E} = -\frac{d\Phi}{dt}$

"Changing $\vec{B}$ field induces an $\vec{E}$ field"

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

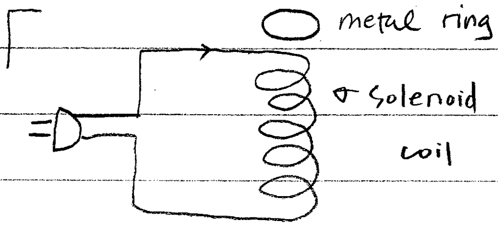
$$\| $$

$$\int \vec{\nabla} \times \vec{E}\, d\vec{a}$$

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Lenz's law : Nature abhors a change in flux
(싫어하다)

Ex 7.6



① switch on.

② what happens to the metal ring?

→ pops up

- The induced $\vec{E}$ field

If $\rho = 0$, we have

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\int (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} \;\; = \;\; \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

Biot-Savart law was

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,') \times \hat{r}}{r^2} d\tau' \quad \text{when } \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

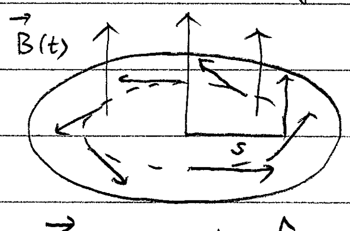
So, we can also say that

$$\vec{E} = -\frac{1}{4\pi} \int \frac{(\partial \vec{B}/\partial t) \times \hat{r}}{r^2} d\tau' \quad \text{when } \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\oint E \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

Faraday's law in integral form.

Ex 7.7) A uniform $\vec{B}(t)$, pointing straight up, fills the shaded circular region. What is the induced $\vec{E}$?



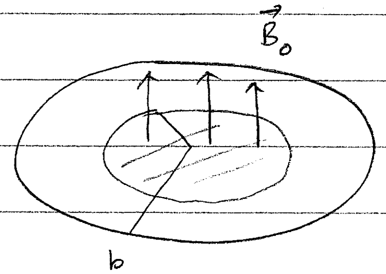
$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$E \cdot 2\pi s = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = \pi s^2 \frac{dB}{dt}$$

$$\vec{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi}$$

Viewed from above, if $\vec{B}$ is increasing, $\vec{E}$ runs clockwise

Ex 7.8)



① A line charge density λ, radius b can freely rotate

② uniform field $\vec{B}_0$ on a region of radius a (< b) vertically up.

③ When $\vec{B}_0$ goes off, angular momentum on the wheel?

각운동량 구하라는데 Rim의 질량이 안주어졌음…

✓ Applying Faraday's law on the rim

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$E\cdot 2\pi b = -\pi a^2 \frac{dB}{dt} \quad \text{or} \quad \vec{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\phi}$$

✓ For a small segment of length $d\vec{l}$, torque?

$$\vec{\tau} = \vec{r}\times\vec{F}$$

$\vec{r}$: b; $\vec{F}$: $qE = \lambda dl E$ $\quad \therefore |\vec{\tau}| = b\lambda E dl$

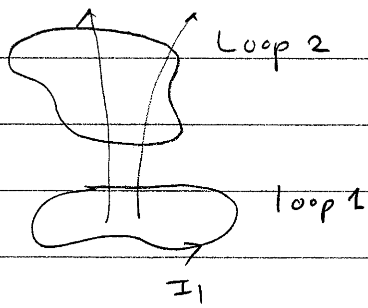
The total torque $N = \oint dN = \oint b\lambda\left(-\frac{a^2}{2b}\right)\frac{dB}{dt}\,dl$

$$= -\frac{\cancel{b}\lambda}{2\cancel{b}} a^2 \frac{dB}{dt}\,\cancel{2}\pi b = -b\pi\lambda a^2\frac{dB}{dt}$$

Angular momentum? $\quad \frac{d\vec{L}}{dt} = \vec{N}$

$$\int N\,dt = -\lambda\pi a^2 b\int_0^{t_{off}}\left(\frac{dB}{dt}\right)dt = -\lambda\pi a^2 b\int_{B_0}^{0} dB = \lambda\pi a^2 b B_0$$

7.2.3. Inductance $\vec{B}_1$



· steady current I_1 at loop 1 produces $\vec{B}_1$

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \int \frac{d\vec{l}_1\times\hat{\mathscr{r}}}{\mathscr{r}^2} \propto I_1$$

· Flux through Loop 2

$$\Phi_2 = \int \vec{B}_1\cdot d\vec{a}_2 \propto I_1$$

$$\equiv M_{21} I_1$$

M_{21}: mutual inductance of the two loops

Now,

$$\Phi_2 = \int_S \vec{B}_1\cdot d\vec{a}_2 = \int_S (\vec{\nabla}\times\vec{A}_1)\cdot d\vec{a}_2 = \oint_\Gamma \vec{A}_1\cdot d\vec{l}_2$$

We had $\vec{A}_1 = \frac{\mu_0 I_1}{4\pi}\oint_\Gamma \frac{d\vec{l}_1}{\mathscr{r}} \rightarrow \Phi_2 = \frac{\mu_0 I_1}{4\pi}\oint_\Gamma\left(\oint_\Gamma \frac{d\vec{l}_1}{\mathscr{r}}\right)\cdot d\vec{l}_2$

We can conclude that

$$M_{21} = \frac{\mu_0}{4\pi} \oint_\Gamma \oint_\Gamma \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r}$$: Neumann formula (not very useful for real calculation...)

1. M_{21} is a (purely) geometrical quantity

2. We can repeat the computation w/ $1 \leftrightarrow 2$

$$\therefore M_{21} = M_{12} \equiv M$$

Ex 7.10) A short solenoid (length l, radius a, n_1 turns/length) in a very long solenoid (radius b, n_2 turns/length). $M = ?$

Think of a current I at outer solenoid:

$$B = \mu_0 n_2 I$$

→ flux on a single loop of short solenoid

$$\Phi_{short}^{single} = B\pi a^2 = \mu_0 n_2 I \pi a^2$$

→ total flux

$$\Phi_{short}^{total} = n_1 l \times \Phi_{short}^{single} = \underbrace{\mu_0 \pi a^2 n_1 n_2 l}_{M} I$$

$$\therefore M = \mu_0 \pi a^2 n_1 n_2 l$$

Suppose I_1 is varied in loop 1. Φ_2 will be changed.

→ emf is induced in loop 2

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -\frac{d}{dt}(MI_1) = -M\frac{dI_1}{dt}$$

⇩

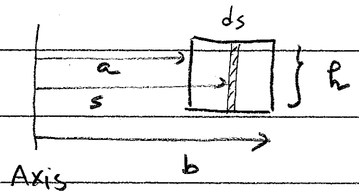
A changing current induces an emf in the source loop itself → induced current $\propto \Phi$

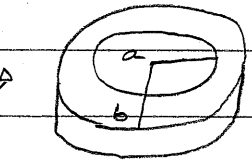
$$\Phi \propto I \equiv LI \rightarrow \mathcal{E} = -L\frac{dI}{dt}$$

L: self inductance (or inductance)

SI unit: $\frac{\text{volt} \cdot \text{second}}{\text{Ampere}} \equiv$ henry (H)

Ex 7.11) L? A toroidal coil with rectangular cross section.





current I

total N turns

The magnetic field B?

$$\oint_{\Gamma} \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} \rightarrow \quad B\cdot 2\pi s = \mu_0 N I \rightarrow B = \frac{\mu_0 N I}{2\pi s}$$

The flux through a single turn:

$$\Phi_s = \int_S \vec{B}\cdot d\vec{a} = \frac{\mu_0 N I}{2\pi} h \int_a^b \frac{1}{s}\, ds = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

Total flux $\Phi_t = \Phi_s \times N = \frac{\mu_0 N^2 I h}{2\pi} \ln\left(\frac{b}{a}\right)$

$\Phi = LI$

$$\rightarrow L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

• Energy in Magnetic Fields

The work done on a unit charge: $-\mathcal{E}$

$\rightarrow$ total work done / unit time

$$-\mathcal{E} I = \frac{dw}{dt} \rightarrow \frac{dw}{dt} = -\left(-L\frac{dI}{dt}\right) I$$

$\frac{dw}{dt} = LI\frac{dI}{dt}$, $W = \frac{1}{2}LI^2$ is obtained. Now let's express W in terms of $\vec{B}$.

• We had $\Phi = LI$ (flux in a loop)

or $\Phi = \int_S \vec{B}\cdot d\vec{a} = \int_S (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint_{\Gamma} \vec{A}\cdot d\vec{\ell}$

So we have $LI = \oint_{\Gamma} \vec{A}\cdot d\vec{\ell}$

$$W = \frac{1}{2}LI^2 = \frac{1}{2} I \oint_{\Gamma} \vec{A}\cdot d\vec{\ell} = \frac{1}{2}\oint_{\Gamma} \vec{A}\cdot\vec{I}\, d\ell$$

Making it to general volume current: $W = \frac{1}{2}\int_V (\vec{A}\cdot\vec{J})\, d\tau$

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$$

$$W = \frac{1}{2}\int_{\mathcal{V}} (\vec{A}\cdot\vec{J})\,d\tau = \frac{1}{2\mu_0}\int_{\mathcal{V}} \underline{\vec{A}\cdot(\vec{\nabla}\times\vec{B})}\,d\tau$$ is obtained.

Now,

$$\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$\therefore\ \vec{A}\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot\vec{B} - \vec{\nabla}\cdot(\vec{A}\times\vec{B})$$

So,

$$W = \frac{1}{2\mu_0}\left[\int_{\mathcal{V}} B^2\,d\tau - \underbrace{\int_{\mathcal{V}} \vec{\nabla}\cdot(\vec{A}\times\vec{B})\,d\tau}_{=\oint_S (\vec{A}\times\vec{B})\cdot d\vec{a}\ =\ 0 \text{ for larger surface.}}\right]$$

$$\therefore\ W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2\,d\tau$$

(Note: for electrostatic case $W = \frac{1}{2}\epsilon_0 \int E^2\,d\tau$)

7.3 Maxwell's eq.

- Electrodynamics Before Maxwell. (one part is missing)

$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$: Gauss's law

$\vec{\nabla}\cdot\vec{B} = 0$ (no name, no monopole)

$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (Faraday's law)

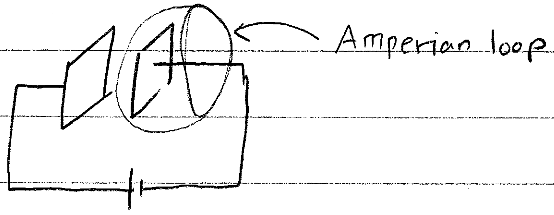
$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$ (Ampère's law)

one trouble: $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{E})}_{=0} = \vec{\nabla}\cdot\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}$

OK!

But... $\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=0} = \mu_0\underbrace{(\vec{\nabla}\cdot\vec{J})}_{\neq 0}$ in general... (true only for steady current)

or. In the process of charging up a capacitor



In integral form, ampere's law reads

$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$I_{enc} = I$ for the right plane

$I_{enc} = 0$? " left balloon-shape surface?

Let's fix it.

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}\left(\epsilon_0 \vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\cdot\left(\epsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

So, if we alter the Ampere's law as

$$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \underbrace{\mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{non-zero only } E=E(t)}$$ then the trouble 1 disappears.

(*) A changing $\vec{E}$ field induces a $\vec{B}$ field.

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}_d$$ displacement current

Now for charging capacitor,

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{Q}{A} \qquad \frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

current in the wire

So, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ becomes

$$\oint_\Gamma \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0\epsilon_0 \int_S \left(\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a}$$

i) flat surface : $E=0$, $I_{enc} = I$

ii) balloon " : $I_{enc}=0$ $\quad \int (\partial\vec{E}/\partial t)\cdot d\vec{a} = \frac{I}{\epsilon_0}$ so

2n term $= \mu_0 I$

so we get the same answer.

733 Maxwell's Equation

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

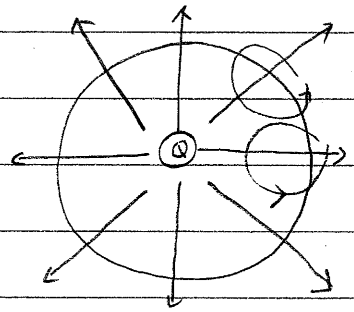
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t}$$

together w/ $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$ they summarize the entire classical electrodynamics.

Special example

Ex) Maxwell eq. (Feynmann, p 18-3)

What happens with a spherically symmetric radial distribution of current. → a sphere with radioactive material on it → a current that is everywhere radially outward.



what is $\vec{B}$?

→ whatever $\vec{B}$ is, it must be Spherically Symmetric.

→ It looks Cancelation is everywhere. Is $\vec{B} = \vec{0}$?

Let's take a look at analytically.

$Q(r)$: total charge inside radius r.

$\vec{J}(r)$: radial current density.

From $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

$$\int \vec{\nabla} \cdot \vec{J} \, d\tau = \oint_S \vec{J} \cdot d\vec{a} = 4\pi r^2 J(r) = \frac{-\partial Q(r)}{\partial t}$$

Now, $\vec{E}$ at radius r : $\frac{Q(r)}{4\pi\epsilon_0} \cdot \frac{1}{r^2}$

$$\therefore \frac{\partial E}{\partial t} = \frac{1}{4\pi\epsilon_0 r^2} \frac{\partial Q}{\partial t} = \frac{1}{4\pi\epsilon_0 r^2} (-4\pi r^2) J(r) = -\frac{J(r)}{\epsilon_0}$$

So, from

$$\vec{\nabla} \times \vec{B} = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t} = \mu_0 J + \mu_0 \epsilon_0 \left(-\frac{J}{\epsilon_0}\right) = 0$$

$\therefore \vec{B} = \vec{0}$!

One can rewrite them as

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \text{LHS} \Rightarrow \text{fields}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \text{RHS} \Rightarrow \text{source}$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0$$

$$\vec{\nabla}\times\vec{B} - \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$

734 Magnetic charge

Let's re-write the Maxwell's eq. in free space (no source)

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

- Now it looks much more symmetric

$$\left.\begin{array}{l}\vec{E} \to \vec{B} \\ \vec{B} \to (-\mu_0\epsilon_0\vec{E})\end{array}\right\} \text{ will make Maxwell's eq invariant !}$$

imagine of

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\epsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\mu_0\vec{J}_m - \frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = \mu_0\rho_m \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}_e + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

ρ_m : magnetic charge

$\vec{J}_m$: " current

then $\vec{\nabla}\cdot\vec{J}_m = -\frac{\partial\rho_m}{\partial t}$ and $\vec{\nabla}\cdot\vec{J}_e = \frac{\partial\rho_e}{\partial t}$ be ok.

But nature dislikes magnetic charge!

735 Maxwell's Eq. in Matter

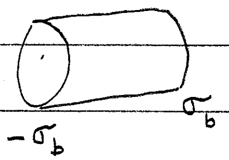
In the static case

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

and $\vec{J}_b = \vec{\nabla}\times\vec{M}$

were introduced. what would be additions for nonstatic case?

Changes in $\vec{P}$ involves a flow of bound charge : $\vec{J}_p$



: a tiny chunk of polarized material

→ polarization introduces a charge density $\sigma_b = P$ at one end and $-\sigma_b$ at the other.

- If P increases a bit → charge on each end increases

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp - \underbrace{\frac{\partial P}{\partial t}}_{\text{current density}} da_\perp$$

$$\therefore \quad \vec{J}_p = \frac{\partial \vec{P}}{\partial t} \quad : \text{polarization current}$$

($\vec{J}_b$: bound current, associated with magnetization of the material)

Let's check $\vec{J}_p = \frac{\partial \vec{P}}{\partial t}$ satisfies the continuity eq.

$$\vec{\nabla} \cdot \vec{J}_p = \vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{P}) = -\frac{\partial \rho_b}{\partial t}$$

$$\therefore \quad \vec{\nabla} \cdot \vec{J}_p + \frac{\partial \rho_b}{\partial t} = 0$$

So, the total charge density : $\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P}$

the " current density

$$\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p$$
$$= \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}.$$

- Gauss's law

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}(\rho_f - \vec{\nabla} \cdot \vec{P})$$

$$\to \epsilon_0 \vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{P} = \rho_f \qquad \vec{\nabla} \cdot \big(\underbrace{\epsilon_0 \vec{E} + \vec{P}}_{\equiv \vec{D}\ :\ \text{electric displacement}}\big) = \rho_f$$

$$\therefore \quad \vec{\nabla} \cdot \vec{D} = \rho_f$$

- Ampère's law

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \left(\vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\to \vec{\nabla} \times \big(\underbrace{\vec{B} - \mu_0 \vec{M}}_{\equiv \mu_0 \vec{H}} \big) = \mu_0 \vec{J}_f + \mu_0 \frac{\partial}{\partial t} \big(\underbrace{\vec{P} + \epsilon_0 \vec{E}}_{\vec{D}} \big)$$

$$\therefore \quad \vec{\nabla}\times\vec{H} = J_f + \frac{\partial \vec{D}}{\partial t}$$

So,

$$\vec{\nabla}\cdot\vec{D} = \rho_f \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

are a set of Maxwell's equations in terms of free charges and currents.

And $\vec{P} = \epsilon_0 \chi_e \vec{E}$ and $\vec{M} = \chi_m \vec{H}$ are satisfied for linear media.

(or $\vec{D} = \epsilon\vec{E}$, $\vec{H} = \frac{1}{\mu}\vec{B}$, with $\epsilon = \epsilon_0(1+\chi_e)$, $\mu = \mu_0(1+\chi_m)$)

Note : $\vec{D}$ is called electric displacement and it make sense because

$$\vec{\nabla}\times\vec{H} = \vec{J}_f + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\vec{J}_d}$$

7.3.6 Boundary Conditions

In general, $\vec{E}, \vec{B}, \vec{D}$, and $\vec{H}$ will be discontinuous at a boundary btw two media, or at $\sigma \neq 0$ or $\vec{K} \neq 0$ surface. Basically

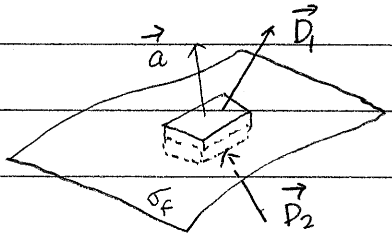
i) $\oint_S \vec{D}\cdot d\vec{a} = Q_{f\,enc}$

ii) $\oint_S \vec{B}\cdot d\vec{a} = 0$

iii) $\oint_\Gamma \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\oint \vec{B}\cdot d\vec{a}$

iv) $\oint_\Gamma \vec{H}\cdot d\vec{l} = I_{f\,enc} + \frac{d}{dt}\oint \vec{D}\cdot d\vec{a}$

will give all conditions needed.



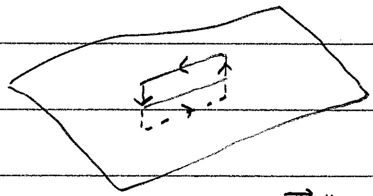
For i)

$$\vec{D}_1 \cdot \vec{a} - \vec{D}_2 \cdot \vec{a} = \sigma_f a$$

or $D_1^{\perp} - D_2^{\perp} = \sigma_f$

For ii), $B_1^{\perp} - B_2^{\perp} = 0$

Now for (iii)



$$\vec{E}_1 \cdot \vec{l} - \vec{E}_2 \cdot \vec{l} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a}$$

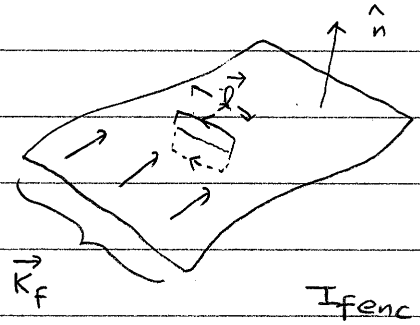
P89 In the limit as the width of loop goes

Fig. 2.37 to zero, RHS $\to 0$.

$$\therefore \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$$

Same way, for iv),

$$\vec{H}_1 \cdot \vec{l} - \vec{H}_2 \cdot \vec{l} = I_{fenc}$$



- No volume current will contribute in the limit of infinitesimal width.

$\longrightarrow$ surface current can!

$$I_{fenc} = \vec{K}_f \cdot (\hat{n} \times \vec{l}) = (\vec{K}_f \times \hat{n}) \cdot \vec{l}$$

$$\therefore \vec{H}_1^{\parallel} - \vec{H}_2^{\parallel} = \vec{K}_f \times \hat{n}$$

In case of linear media, one can re-write

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f \qquad \vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$$