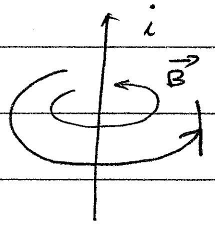
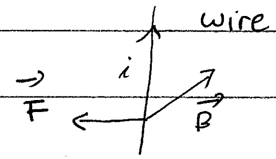
Magnetostatics

- Stationary charge produces $\vec{E}$!
- (? magnetic charge ? (= monopole?)) " $\vec{B}$?

ANS: current (or moving charges)





($\vec{F} = i \times \vec{B}$)

(in physics101)

Magnetic Forces

Magnetic force on a charge Q, moving $\vec{v}$ in $\vec{B}$:

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$$

: Lorentz force law

unit:
SI unit

$$[N] = [C]\frac{[m]}{[s]} \cdot [T] \rightarrow [T] = [N]\frac{[s]}{[C \cdot m]}$$

Q: where is Lorentz force law from? → axiom? empirical?

Anyhow, in general,

$$\vec{F} = Q(\vec{E} + \vec{v} \times \vec{B})$$

one can add axial (= pseudo) to normal vector!

- Magnetic forces do no work

∵ If Q moves $d\vec{l} = \vec{v}\,dt$, the work done:

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt = 0$$

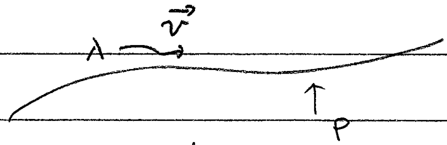
— $\vec{F}_{mag}$ can change direction of the charged ptl but not amount of the speed.

5.1.3 Currents

Current: charge per unit time

$$1\text{ A} = 1\text{ C/s}$$

Suppose a line charge density λ travels.



- charge passed during the time window Δt : Δq " $\lambda v \Delta t$
- Current ? $\vec{I} = \dfrac{\lambda \vec{v} \Delta t}{\Delta t} = \lambda \vec{v}$

(Griffiths claims current is a vector !?)

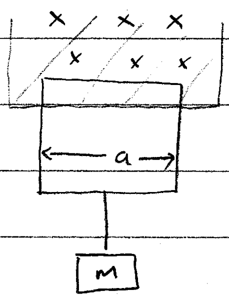
So, $\Delta \vec{F}_{mag} = \Delta q\, \vec{v} \times \vec{B}$ and the total force due to on a segment of current-carrying wire is

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq = \int (\vec{v} \times \vec{B}) \lambda\, dl = \int (\vec{I} \times \vec{B})\, dl$$

In magnetostatics, $\vec{I}\, dl = I\, d\vec{l}$ with I : constant along the wire

$$\vec{F}_{mag} = I \int d\vec{l} \times \vec{B} \quad \text{is satisfied.}$$

Ex) 5.3



For what current I in the loop the magnetic force upward balance gravitational force ?

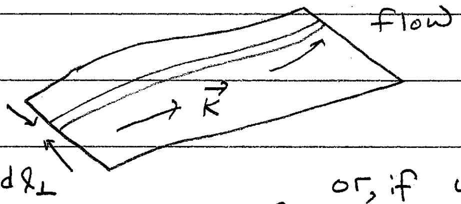
i) We need upward $\vec{F}_{mag}$. → $\vec{I}$ must be clockwise.

ii) $F_{mag} = mg$
$F_{mag} = IBa$ → $I = \dfrac{mg}{ba}$

Q: If one increases I, m will move up.
So $\vec{B}$ did some work ?

↳ interesting discussion in page 218

· Surface current density $\vec{K}$: current per unit width



$$\vec{K} \equiv \frac{d\vec{I}}{d\ell_\perp}$$

or, if we use surface charge density σ,

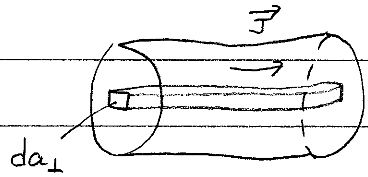
$$K = \frac{d}{d\ell_\perp}\frac{\overbrace{\lambda\, d\ell}^{q}}{dt} = \underbrace{\frac{d\lambda}{d\ell_\perp}}_{\sigma} v \quad \therefore \quad \vec{K} = \sigma \vec{v}$$

The magnetic force on the surface current is

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, \sigma\, da = \int (\vec{K} \times \vec{B})\, da$$

· Now for the 3-D case:

Volume current density $\vec{J}$



$$\vec{J} \equiv \frac{d\vec{I}}{da_\perp}$$: $\vec{J}$ is the current per unit area

If we introduce the volume charge density ρ

$\vec{J} = \rho \vec{v}$ is obtained.

Now the surface integral is for closed surface

Total current $I = \oint_S J\, da_\perp = \oint_S \vec{J} \cdot d\vec{a}$

or, using divergence theorem,

$$\oint_S \vec{J} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{J})\, d\tau$$ is satisfied.

Q_{inside} S

한편, $\oint \vec{J} \cdot d\vec{a} = -\frac{dQ_{inside}}{dt}$ is in general true.

$$= -\frac{d}{dt}\int_V \rho\, d\tau$$

$\therefore$ $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ and states conservation of charge.

: called continuity equation

5.2 The Biot-Savart Law

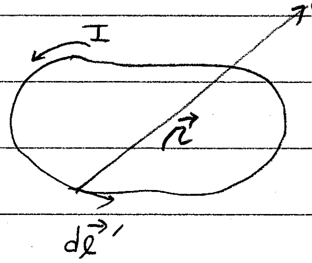
- Stationary charges $\rightarrow$ constant $\vec{E}$: electrostatics
- steady currents $\rightarrow$ " $\vec{B}$: magnetostatics

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \vec{J}}{\partial t} = \vec{0}$$

$\rightarrow$ So $\vec{\nabla} \cdot \vec{J} = 0$ as well.

- The magnetic field of a steady current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \vec{\mathscr{r}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times \vec{\mathscr{r}}}{\mathscr{r}^2}$$



: the integration is along the current path.

μ_0 : permeability of free space

$4\pi \times 10^{-7}\ N/A^2$

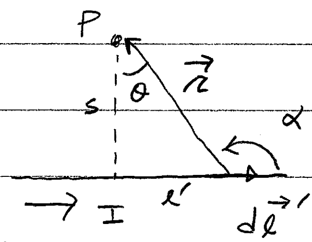
Unit of $\vec{B}$?

$$[B] = \underbrace{\frac{N}{A^2}}_{\mu_0} \cdot \underbrace{\frac{A}{m}}_{I,\ \frac{dl}{\mathscr{r}^2}} = N/(A\cdot m) \equiv 1\,T$$

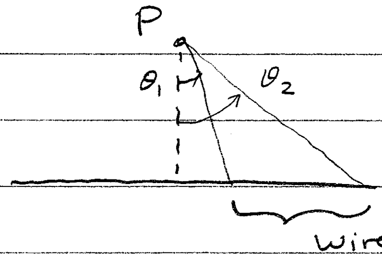
$\uparrow$ tesla

(1 tesla $= 10^4$ gauss)

Ex 5.5) Find $\vec{B}$ a distance s ~~from~~ a long straight wire carrying I.



and



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{\mathscr{r}}}{\mathscr{r}^2}$$

① $|d\vec{l}' \times \hat{\mathscr{r}}| = |\, dl' \sin\alpha\,| = dl' \cos\theta$

② also $l' = s\tan\theta$ gives us $dl' = s\sec^2\theta\, d\theta = s \cdot \frac{1}{\cos^2\theta}\, d\theta$

③ $s = \mathscr{r}\cos\theta$, " $\frac{1}{\mathscr{r}^2} = \frac{\cos^2\theta}{s^2}$

Putting ①-③ together,

$$B = \frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2} \underbrace{\frac{s}{\cos^2\theta}\,d\theta\cdot\cos\theta}_{dl'\cos\theta}\cdot\underbrace{\frac{\cos^2\theta}{s^2}}_{1/r^2}$$

$$= \frac{\mu_0 I}{4\pi s}\int_{\theta_1}^{\theta_2}\cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)$$

In case of infinite wire, $\theta_1 = -\pi/2$, $\theta_2 = \pi/2$

$$B = \frac{\mu_0 I}{2\pi s}$$ (What was E for infinitely long charged wire? $E = \frac{\lambda}{2\pi\epsilon_0}\cdot\frac{1}{r}$) same behavior.

· Direction? $d\vec{l}' \times \hat{r} \parallel \hat{\phi}$

$$\therefore \vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

· Force/length between two long, parallel wires (distance d) with currents I_1, I_2: B field at (2) due to (1)

$$B = \frac{\mu_0 I_1}{2\pi d}, \quad \vec{F} = I\int d\vec{l}\times\vec{B}$$ in general, so

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl$$

$$f \equiv F/\int dl = \frac{\mu_0}{2\pi}\cdot\frac{I_1 I_2}{d}$$ (attractive?!)

5.3 The divergence and curl of B

• For the straight-line current, let's compute

$$\underset{\substack{\uparrow\\ \text{circular path}\\ \text{of radius } s}}{\oint \vec{B}\cdot d\vec{l}} = \oint \frac{\mu_0 I}{2\pi s}\hat{\phi}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi s}\underbrace{\oint dl}_{2\pi s} = \mu_0 I$$

$\hat{\phi}\cdot d\vec{l} = dl$

$\mu_0 I$ independent of s!

And it is true for ANY closed path!

$$d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z} \qquad \text{(in cylindrical coord.)}$$

$$\hat{\phi}\cdot d\vec{l} = s\,d\phi$$

$$\therefore \oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s}\, s\,d\phi = \mu_0 I$$

↑ any closed path

In general, $\oint \vec{B}\cdot d\vec{l} = \mu_0 \underbrace{I_{enc}}_{\text{total current enclosed by the loop.}}$

i) Stokes' theorem:

$$\int (\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \oint \vec{A}\cdot d\vec{l}$$

ii) $I_{enc} = \int \vec{J}\cdot d\vec{a}$

i) + ii) gives us

$$\oint \vec{B}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 I_{enc}$$
$$= \int \vec{J}\cdot d\vec{a}$$

$$\therefore \vec{\nabla}\times\vec{B} = \mu_0 I_{enc}$$

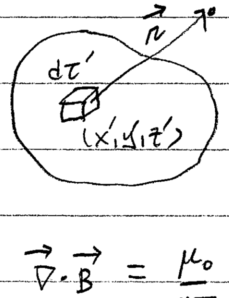
- The divergence and curl of $\vec{B}$

The Biot-Savart law for the general volume current

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \underbrace{\frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}}_{\frac{1}{l^2}\left(\frac{\text{current}}{l^2}\right)\cdot l^3} d\tau' \qquad \left(\text{wire}: \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \underbrace{\frac{\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl'}_{\frac{\text{current}}{l^2}\, l}\right)$$

이러한 extension
쉽게 guess 하는 것도 중요!

$\Rightarrow \frac{\text{current}}{l}$ ← same → current/l



- $\vec{\mathscr{r}} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$
- $d\tau' = dx'\,dy'\,dz'$
- Integration is over the primed coord.

Now, $\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) d\tau'$

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\left(\vec{\nabla}\times\vec{J}\right) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

But $\vec{\nabla}\times\vec{J} = 0$ since $\vec{J} = \vec{J}(\vec{r}')$ only

$\vec{\nabla}\times\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 0$ (problem 1.63)

$\vec{\nabla} = \vec{\nabla}(\vec{r})$, $\hat{\mathscr{r}} = \hat{\mathscr{r}}(\vec{r} - \vec{r}')$

$\therefore$ $\vec{\nabla}\cdot\vec{B} = 0$ is satisfied.

• $\vec{\nabla}\times\vec{B}$ when $\vec{J}$ is present:

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int \vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

$$\vec{\nabla}\times\left(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{J}\left(\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) - \left(\vec{J}\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

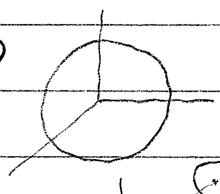
from $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$

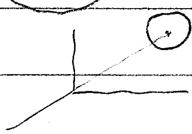
$A \to \vec{J}$ $\qquad (\vec{B}\cdot\vec{\nabla})\vec{A} = 0$, $\quad \vec{\nabla}\cdot\vec{A} = 0$

The 1st term:

$$\vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = ? \qquad \text{ANS: } 4\pi\,\delta^3(\vec{\mathscr{r}})$$

$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$ $\qquad \int \vec{\nabla}\cdot\left(\frac{1}{4\pi\epsilon_0}\frac{q\,\hat{r}}{r^2}\right)d\tau = q/\epsilon_0$

i)  : $\int \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = 4\pi$

ii)  : $\int \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)d\tau = 0$

$$\therefore \quad \vec{\nabla}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = 4\pi\,\delta(\vec{\mathscr{r}})$$

The 2nd term: $\vec{\nabla}' = -\vec{\nabla}$

$$-\left(\vec{J}\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{J}\cdot\vec{\nabla}'\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

x - component: $\vec{J}\cdot\vec{\nabla}'\left(\frac{x-x'}{\mathscr{r}^3}\right) = \vec{\nabla}'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J} - \frac{(x-x')}{\mathscr{r}^3}\vec{\nabla}'\cdot\vec{J}\right]$

$\vec{\nabla}'\cdot\vec{J} = 0$ for steady current

$$\therefore -(\vec{J}\cdot\vec{\nabla})\frac{\hat{\imath}}{\imath^2}\Big|_x = \vec{\nabla}'\cdot\left[\frac{(x-x')}{\imath^3}\vec{J}\right]$$

$$\rightarrow \int_V \vec{\nabla}'\cdot\left[\frac{(x-x')}{\imath^3}\vec{J}\right]d\tau' = \oint_S \frac{x-x'}{\imath^3}\vec{J}\cdot d\vec{a} = 0$$

for infinitely large surface.

$$\therefore \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}(\vec{r}')\,4\pi\delta^3(\vec{r}-\vec{r}')\,d\tau' = \mu_0\vec{J}(\vec{r})$$

5.3.3 Ampere's law

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

is called Ampere's law. (differential form). Using Stokes' theorem:

$$\int(\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \oint \vec{B}\cdot d\vec{l} = \mu_0\underbrace{\int\vec{J}\cdot d\vec{a}}_{I_{enc}}$$

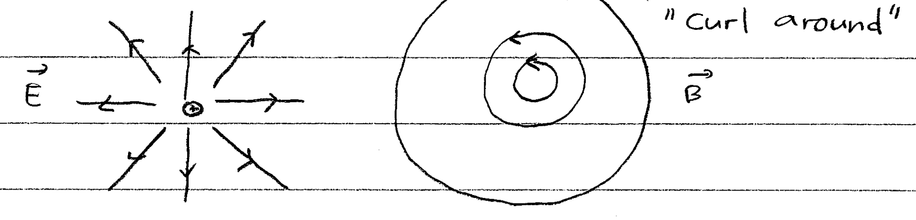
$$\hookrightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

Comparison of Magnetostatic and Electrostatics

$$\begin{cases}\vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0}\\ \vec{\nabla}\times\vec{E} = 0\end{cases} \qquad \begin{cases}\vec{\nabla}\cdot\vec{B} = 0\\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} = \dfrac{1}{\epsilon_0 c^2}\vec{J}\ \text{(Feynman)}\end{cases}$$

$$\sqrt{\mu_0\epsilon_0} = \frac{1}{c}$$

$$\oplus\ \vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$$



- Magnetic effects are attributable to electric charges in motion (see Feynman ch 13-6)

5.4 Magnetic Vector potential

- We had a scalar potential V to replace $\vec{E}$ since $\vec{\nabla}\times\vec{E}=0$

$$\vec{E} = -\vec{\nabla}V \qquad (\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\vec{\nabla}V = 0)$$

- similar mannar, $\vec{\nabla}\cdot\vec{B}=0$ gives us

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad (\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0)$$

Now, $\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J}$

Let's choose $\vec{\nabla}\cdot\vec{A}=0$ (fixing the gauge)

Suppose our original vector potential $\vec{A}_0$ is not divergenceless ($\vec{\nabla}\cdot\vec{A}_0 \neq 0$). We can always add $\vec{\nabla}\lambda$

$\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$ such that

1) $\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\vec{A}_0 + \nabla^2\lambda$ and choose $\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}_0$

So that $\vec{\nabla}\cdot\vec{A}=0$

2) $\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\vec{A}_0 + \underbrace{\vec{\nabla}\times(\vec{\nabla}\lambda)}_{=0}$ so $\vec{B}$ remains same.

So, $\nabla^2\vec{A} = -\mu_0\vec{J}$ is satisfied.

(These are 3 Poission equations!)

Note: in electrostatics,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \longrightarrow V = \frac{1}{4\pi\epsilon_0}\int\frac{\rho}{\mathscr{r}}d\tau'$$

$$\text{so, } \nabla^2\vec{A} = -\mu_0\vec{J} \longrightarrow \vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{\mathscr{r}}d\tau'$$

will automatically satisfied

(For line current

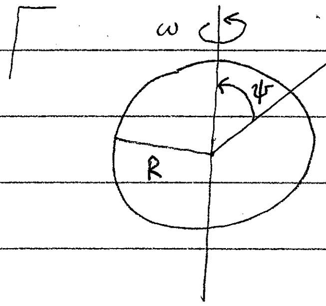
$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{I}}{\mathscr{r}}dl' = \frac{\mu_0 I}{4\pi}\int\frac{1}{\mathscr{r}}d\vec{l}'$$

for surface current

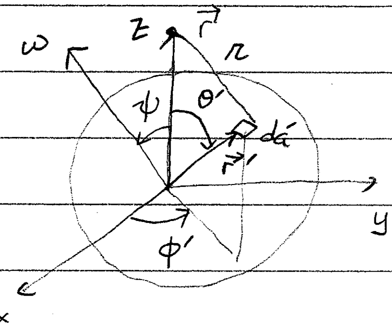
$$\vec{A} = \frac{\mu_0}{4\pi}\int\frac{\vec{K}}{\mathscr{r}}da'$$

Ex 5.11) Very complicated example.

A spherical shell (radius R), uniform surface charge σ spinning at angular velocity $\vec{\omega}$. $\vec{A}$ at $\vec{r}$? $B = ?$



Let's choose
$\vec{r}$ lies on z axis
$\vec{\omega}$ lies on x–z plane



Our formula for $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{K(\vec{r}\,')}{\mathscr{r}}\, da'$

- $\vec{K} = \sigma \vec{v}$
- $\mathscr{r} = [R^2 + r^2 - 2Rr\cos\theta']^{1/2}$ $\quad da' = R^2 \sin\theta'\, d\theta'\, d\phi'$
- $\vec{v} = \vec{\omega} \times \vec{r}\,'$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\left[-(\cos\psi\sin\theta'\sin\phi')\hat{x} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\hat{y} + (\sin\psi\sin\theta'\sin\phi')\hat{z}\right]$$

For the ϕ' integral, only will be nonzero.

$$\left(\because \int_0^{2\pi} \sin\phi'\, d\phi' = \int_0^{2\pi} \cos\phi'\, d\phi' = 0\right)$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 \sigma}{4\pi} \int \frac{(-)\, R\omega\sin\psi\cos\theta'}{[R^2 + r^2 - 2Rr\cos\theta']^{1/2}} R^2 \sin\theta'\, d\theta'\, d\phi' \;\hat{y}$$

$$= -\frac{\mu_0 R^3 \sigma\omega\sin\psi}{2}\left[\int_0^{\pi} \frac{\cos\theta'\sin\theta'\, d\theta'}{[R^2 + r^2 - 2Rr\cos\theta']^{1/2}}\right]\hat{y}$$

Let $u \equiv \cos\theta'$, $du = -\sin\theta'\, d\theta'$ and the integral becomes

$$\int_1^{-1} \frac{u\, du}{\sqrt{R^2 + r^2 - 2Rru}}$$

($y = a + bx$, $x = \frac{y-a}{b}$ $\quad dy = b\, dx$

Note: $$\int \frac{x\, dx}{\sqrt{a+bx}} = \int \left(\frac{y-a}{b}\right) y^{-1/2} \frac{dy}{b} = \frac{1}{b^2}\int \left(y^{1/2} - a y^{-1/2}\right) dy$$

$$= \frac{1}{b^2}\left(\frac{2}{3}y^{3/2} - 2ay^{1/2}\right) = \frac{2y^{1/2}}{b^2}\left(\frac{1}{3}y - a\right)$$

$$= \frac{2\sqrt{a+bx}}{3b^2}\left(a+bx-3a\right) = \frac{2\sqrt{a+bx}}{3b^2}\left(bx-2a\right)$$

$$\therefore \int_1^{-1} \frac{u\,du}{\sqrt{R^2+r^2-2Rru}} = \frac{\cancel{2}\sqrt{R^2+r^2-2Rru}}{3\cdot \cancel{4}R^2r^2}\left(-\cancel{2}Rru - \cancel{2}(R^2+r^2)\right)\Bigg|_1^{-1}$$

$$= \frac{1}{3R^2r^2}\left[-(R+r)(R^2+r^2-Rr) + |R-r|(R^2+r^2+Rr)\right]$$

i) $R>r \rightarrow \frac{1}{3R^2r^2}\left[-(R+r)(R^2+r^2-Rr) + (R-r)(R^2+r^2+Rr)\right]$

$$= \frac{1}{3R^2r^2}\left[\begin{array}{l} \cancel{-R^3} - \cancel{Rr^2} + R^2r - rR^2 - r^3 + \cancel{Rr^2} \\ \cancel{+R^3} \cancel{+Rr^2} + R^2r - rR^2 - r^3 - \cancel{Rr^2} \end{array}\right]$$

$$= -\frac{2r}{3R^2}$$

ii) $R<r \rightarrow \frac{1}{3R^2r^2}\left[-(R+r)(R^2+r^2-Rr) - (R-r)(R^2+r^2+Rr)\right]$

$$= \frac{1}{3R^2r^2}\left[\begin{array}{l} -R^3 - \cancel{Rr^2} + R^2r - rR^2 - r^3 + \cancel{Rr^2} \\ -R^3 - \cancel{Rr^2} - R^2r + rR^2 + r^3 + \cancel{Rr^2} \end{array}\right]$$

$$= -\frac{2R}{3r^2}$$

Since $\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ 0 & 0 & r \end{vmatrix} = -r\omega\sin\psi\,\hat{y}$

i) $R>r$ (inside) $\vec{A}(\vec{r}) = \frac{\mu_0 R^3\sigma\,\omega\sin\psi}{2}\left(-\frac{2r}{3R^2}\right)\hat{y}$

$$= \frac{\mu_0 R\sigma}{3}(\vec{\omega}\times\vec{r})$$

ii) $R<r$ (outside) $\vec{A}(\vec{r}) = \frac{\mu_0 R^3\sigma\,\omega\sin\psi}{2}\left(-\frac{2R}{3r^2}\right)\hat{y}$

$$= \frac{\mu_0 R^4\sigma}{3r^3}(\vec{\omega}\times\vec{r})\,!$$

We can write them in usual coord.

$\vec{\omega} \parallel \hat{z}$ $\quad$ $\vec{r}$ is with (r, θ, ϕ)

$$\vec{A}(r,\theta,\phi) = \begin{cases} \dfrac{\mu_0 R \sigma \omega}{3} \, r \sin\theta \, \hat{\phi} & (r < R) \\[2ex] \dfrac{\mu_0 R^4 \sigma \omega}{3} \, \dfrac{\sin\theta}{r^2} \, \hat{\phi} & (r \geq R) \end{cases}$$

$\vec{B} = ?$ $\quad (r < R$, inside$)$

$\vec{B} = \vec{\nabla} \times \vec{A} = ?$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r\sin\theta\, A_\phi \end{vmatrix}$$

$$= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} \frac{\mu_0 R \sigma \omega}{3}$$

$$= \frac{\mu_0 R \sigma \omega}{3 r^2 \sin\theta} \left(2r^2 \sin\theta \cos\theta \, \hat{r} - 2r \sin^2\theta \cdot r\hat{\theta} \right)$$

$$= \frac{2\mu_0 R \sigma \omega}{3} \Big(\underbrace{\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}_{= \hat{z}} \Big) = \frac{2}{3} \mu_0 \sigma R \vec{\omega}$$

: uniform!

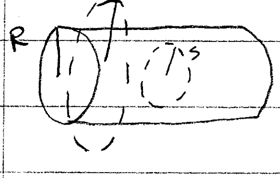
Ex 5.12 $\quad$ A cute trick to get $\vec{A}$

$\vec{A}$ of infinite solenoid with n turns

Direct integration impossible since it is infinite.

So, let's try

$$\oint \vec{A} \cdot d\vec{l} = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \text{magnetic flux } \Phi$$



$s \leq R$: $\displaystyle \oint \vec{A} \cdot d\vec{l} = A \, 2\pi s = \int \vec{B} \cdot d\vec{a} = \mu_0 n i \, \pi s^2$

$$\vec{A} = \frac{\mu_0 n i}{2} s \hat{\phi}$$

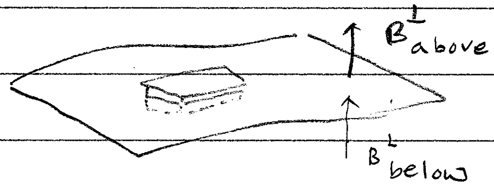
(why $\hat{\phi}$?)

$$\vec{\nabla} \times \vec{\phi} = \frac{1}{s} \begin{vmatrix} \hat{\rho} & s\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \cdots \end{vmatrix} = \frac{1}{s} \cdot s \hat{\phi} = \hat{\phi} \; !$$

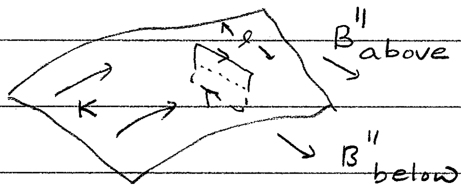
5.4.2 Boundary Conditions

- $\vec{\nabla}\cdot\vec{B}=0 \rightarrow \int \vec{\nabla}\cdot\vec{B}\,d\tau' = \oint \vec{B}\cdot d\vec{a} = 0$

$\Rightarrow B^{\perp}_{above} = B^{\perp}_{below}$ is obtained.



- For the tangential components



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$
$$= \mu_0 K l$$
$$= (B^{\parallel}_{above} - B^{\parallel}_{below})\, l$$
$$\therefore\ B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

$\therefore \left(\begin{array}{l} \vec{B} \text{ parallel to surface} \\ \text{but perpendicular to } \vec{I} \end{array}\right) \rightarrow$ discontinuous, by $\mu_0 K$

$$\therefore\ \vec{B}_{above} - \vec{B}_{below} = \mu_0 (K \times \hat{n})$$

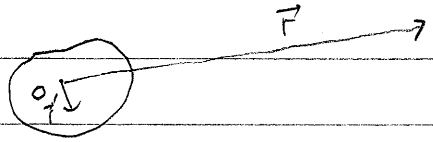
Similar to electric potential, the vector potential is continuous across any boundary

$$\vec{A}_{above} = \vec{A}_{below}$$

5.4.3 Multiple Expansion of the Vector potential

We learned that

$$\frac{1}{r\!\!\!/} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$



So $\left(\frac{r'}{r}\right) \ll 1$!

α: angle btw $\vec{r}$ and $\vec{r}'$

$\vec{A}$ of a current loop is

So, $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\oint \frac{1}{r\!\!\!/}\, d\vec{l}' = \frac{\mu_0}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$

monopole $P_0 = 1$ dipole $P_1(x) = x$ $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ quadrupole

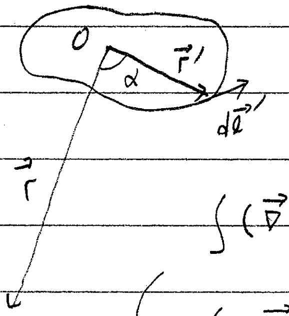
$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r' \cos\alpha \, d\vec{l}' + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)d\vec{l}' + \cdots\right]$$

- Magnetic monopole term: always zero

$$\oint d\vec{l}' = 0 \quad \text{by calculation} \quad (\text{or } \vec{\nabla}\cdot\vec{B} = 0)$$

- Dipole term:

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint (r') \cos\alpha \, d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')\, d\vec{l}'$$



Now, let's look at $\oint (\hat{r}\cdot\vec{r}')\, d\vec{l}'$:

From Stoke's theorem,

$$\int (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint \vec{v}\cdot d\vec{l} \qquad \text{Let's put } \vec{v} = \vec{c}\,T$$

$\vec{c}$: constant vector

$$\int \vec{\nabla}\times(\vec{c}\,T)\cdot d\vec{a} = \oint \vec{c}\,T\cdot d\vec{l} = \vec{c}\cdot\oint T\, d\vec{l}$$

LHS:

$$= \int T\underbrace{(\vec{\nabla}\times\vec{c})}_{=0}\cdot d\vec{a} - \int \vec{c}\times(\vec{\nabla}T)\cdot d\vec{a}$$

$$= -\int \vec{c}\cdot\left[(\vec{\nabla}T)\times d\vec{a}\right]$$

$$\therefore \int \vec{\nabla}T\times d\vec{a} = -\oint T\, d\vec{l} \quad \text{or}$$

$$\int \vec{\nabla}'T\times d\vec{a}' = -\oint T\, d\vec{l}' \qquad \text{if } T = \hat{r}\cdot\vec{r}'$$

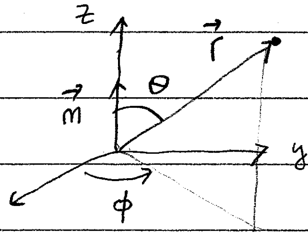
$$\vec{\nabla}'(\hat{r}\cdot\vec{r}') = \hat{r}$$

$$\text{so } -\hat{r}\times\int d\vec{a}' = \oint (\hat{r}\cdot\vec{r}')\, d\vec{l}'$$

$$\therefore \vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\int d\vec{a}'\times\hat{r} \qquad \text{If we define } \vec{m} \equiv I\int d\vec{a} = I\vec{a}$$

$$= \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} \qquad \vec{m}: \text{magnetic dipole moment}$$

To calculate $\vec{B}$ from $\vec{A}$ (easily), let's choose the following coord.:



$$\vec{m}\times\hat{r} = (m\hat{z})\times\hat{r}$$
$$= (m)(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\times\hat{r}$$
$$= m\sin\theta\,\hat{\phi}$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla}\times\vec{A} = ?$$

Note that $\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix}$

so, $\vec{\nabla}\times\vec{A} = \frac{1}{r^2\sin\theta}\cdot\frac{\mu_0\cdot m}{4\pi}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ 0 & 0 & \frac{\sin^2\theta}{r} \end{vmatrix}$

$$= \frac{1}{r^2\sin\theta}\frac{\mu_0 m}{4\pi}\left(\hat{r}\cdot\frac{2\sin\theta\cos\theta}{r} + \hat{\theta}\,\frac{\sin^2\theta}{r}\right)$$

$$= \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

Same as $\vec{E}_{dip}$ in p158! (dipole for $\vec{E}$)