4. Electric Fields in Matter

- conductors
- insulators (or dielectrics)

External electric field can induce dipole on neutral atoms

$$\vec{p} = \alpha \vec{E}$$

$\vec{p}$: dipole moment $\vec{E}$: external electric field α: atomic polarizability

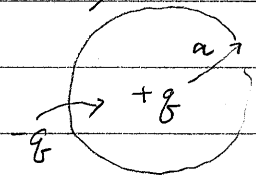
Unit of α?

$$\vec{p} \equiv \int \vec{r}' \rho(\vec{r}')\, d\tau' \qquad [p] = C \cdot m$$

$$\rightarrow \quad C \cdot m = [\alpha] \cdot \frac{1}{4\pi\epsilon_0} \frac{C}{m^2} \qquad \therefore \quad \frac{[\alpha]}{4\pi\epsilon_0} = m^3$$

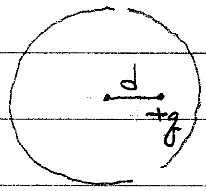
Ex) $\frac{\alpha(H)}{4\pi\epsilon_0} = 0.667 \times 10^{-30}\ m^3$

Ex 4.1) A primitive model for an atom



✓ a point nucleus ($+q$)

✓ surrounded by a uniformly charged spherical cloud ($-q$) of radius a.



External field = field due to $-q$ pulling $+q$ to the left

$$\rightarrow \vec{E}_e$$

$$\uparrow\ E = \frac{1}{4\pi\epsilon_0} \frac{q'}{d^2} = \frac{1}{4\pi\epsilon_0} \frac{q}{d^2} \cdot \frac{d^3}{a^3} = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3}$$

$$\therefore\ E_e = \frac{1}{4\pi\epsilon_0} \frac{qd}{a^3} = \frac{1}{4\pi\epsilon_0} \frac{p}{a^3}$$

$$\therefore\ \alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v$$

v: volume of the atom ($v = \frac{4\pi}{3} a^3$)

$$\frac{\alpha}{4\pi\epsilon_0} = a^3 \approx (10^{-10}\ m)^3 = 10^{-30}\ m!$$

(not bad!?)

For real molecules, the atomic polarizability depends on the direction of $\vec{E}$ field. So the most general sol. is

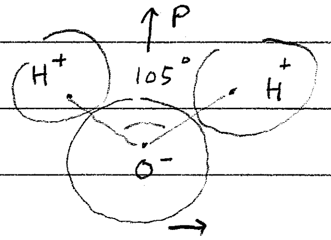
$$p_x = \alpha_{xx} E_x + \alpha_{xy} E_y + \alpha_{xz} E_z$$

$$p_y = \alpha_{yx} E_x + \alpha_{yy} E_y + \alpha_{yz} E_z$$

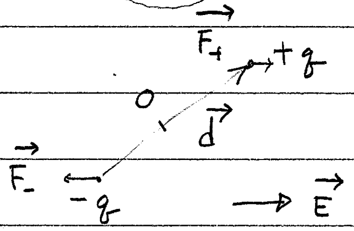
$$p_z = \alpha_{zx} E_x + \alpha_{zy} E_y + \alpha_{zz} E_z$$

· Alignment of polar Molecules

H_2O



What happens if a polar molecule is placed in an electric field?



· Forces cancel exactly

· There will be a torque. $(\vec{N})$

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F})$$
$$= (\vec{d}/2) \times (q\vec{E}) + (-\vec{d}/2) \times (-q\vec{E}) = q\,\vec{d} \times \vec{E}$$
$$= \vec{p} \times \vec{E}$$

Non-uniform field? $\longrightarrow$ There will be net force

$$\vec{F} = \vec{F}_+ + \vec{F}_- = q\,(\vec{E}_+ - E_-) = q\,\Delta\vec{E}$$

$\Delta E_x = (\vec{\nabla} E_x) \cdot \vec{d}$ and so on:

$\therefore\ \Delta\vec{E} = (\vec{d} \cdot \vec{\nabla})\,\vec{E}$ (change of $\vec{E}$ in the direction of $\vec{d}$)

$$\therefore\ \vec{F} = (\vec{p} \cdot \vec{\nabla})\,\vec{E}$$

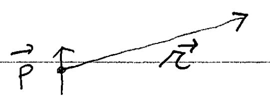
4.1.4 Polarization

Polarization $\vec{P} \equiv$ dipole moment per unit volume

Suppose we have polarized material. Q: what is the field due to this object?

· For a single dipole,

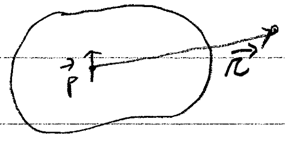
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{\imath}}{\imath^2}$$

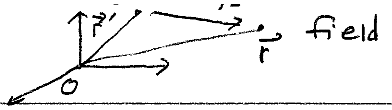


· For continuous distribution,

$\vec{P} = \dfrac{\vec{p}}{d\tau'}$ ← volume element

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(\vec{r}') \cdot \hat{\imath}}{\imath^2}\, d\tau'$$





(p15 pro. 1.13)

Since $\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$ (differentiation is w.r.t. the source coord $\vec{r}'$)

$$\left[\ \underbrace{\vec{\nabla}'\frac{1}{\mathscr{r}} = ?}_{\text{x-comp of}}\quad \vec{\nabla}'\frac{1}{\mathscr{r}} = \frac{\partial}{\partial x'}\left[(x'-x)^2+(y'-y)^2+(z'-z)^2\right]^{-1/2}\right.$$

$$= (-\tfrac{1}{2})(\mathscr{r}^2)^{-3/2}\cdot 2(x'-x) = \frac{1}{\mathscr{r}^3}(x'-x)$$

$$\therefore\ \vec{\nabla}'\frac{1}{\mathscr{r}} = \frac{1}{\mathscr{r}^3}(x'-x)\,\hat{x} + \frac{1}{\mathscr{r}^3}(y'-y)\,\hat{y} + \frac{1}{\mathscr{r}^3}(z'-z)\,\hat{z}$$

$$\left. = \frac{\vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\ \right]$$

$$V = \frac{1}{4\pi\epsilon_0}\int_v \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\int_v \vec{\nabla}'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right)d\tau' - \int_v \frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})\,d\tau'\right]$$

$$= \frac{1}{4\pi\epsilon_0}\left[\underbrace{\oint_S \frac{1}{\mathscr{r}}\vec{P}\cdot d\vec{a}'}_{\text{potential of a surface charge}} - \underbrace{\int_v \frac{1}{\mathscr{r}}(\vec{\nabla}'\cdot\vec{P})\,d\tau'}_{\text{potential of a volume charge}}\right]$$

Define a surface charge, a volume charge

$$\underbrace{\sigma_b}_{\text{bound}} \equiv \vec{P}\cdot\underbrace{\hat{n}}_{\text{normal unit vector}} \qquad \rho_b \equiv -\vec{\nabla}'\cdot\vec{P}$$

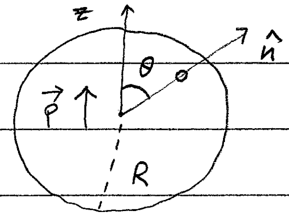
$$V(r) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{\mathscr{r}}\,da' + \frac{1}{4\pi\epsilon_0}\int_v \frac{\rho_b}{\mathscr{r}}\,d\tau'$$

$\therefore$ Potential (and $\vec{E}$) of polarized object

= from $\rho_b = -\vec{\nabla}'\cdot\vec{P}$

and from $\sigma_b = \vec{P}\cdot\hat{n}$

Ex 4.2) $\vec{E}$ by a uniformly polarized sphere of Radius R ?



$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0$ (uniformly polarized)

$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$

From Ex 3.9), we know that

$$V(r,\theta) = \begin{cases} \dfrac{P}{3\epsilon_0}\, r\cos\theta & r<R \\ \dfrac{P}{3\epsilon_0}\dfrac{R^3}{r^2}\cos\theta & r \geq R \end{cases}$$

Since $r\cos\theta = z$

$$\vec{E} = -\vec{\nabla}V = -\frac{P}{3\epsilon_0}\hat{z} = -\frac{1}{3\epsilon_0}\vec{P} \quad \text{for } r<R$$

For $r>R$,

$$V(r,\theta) = \frac{P}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta$$

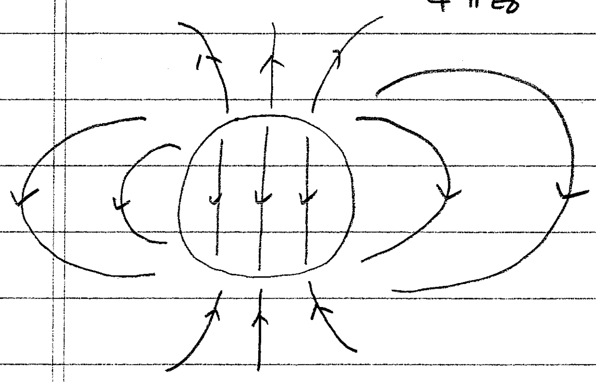
Note that $V \propto \frac{1}{r^2}$ (same as for the perfect dipole case)

So, if we put $\vec{p} = \frac{4}{3}\pi R^3\,\vec{P}$ (polarization = $\frac{\text{dipole}}{\text{Volume}}$)

$$V = \frac{1}{3\epsilon_0}\cdot\frac{3}{4\pi R^3}\cdot\frac{p\,R^3}{r^2}\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\vec{p}\cdot\hat{r}}{r^2}$$

and is identical to the potential for a perfect dipole at the origin.



4.3 The Electric Displacement

— Gauss's law in the presence of Dielectrics

· Polarization produces accumulation of bound charge

$$\begin{cases} \rho_b = -\vec{\nabla}\cdot\vec{P} & \text{within the dielectric} \\ \sigma_b = \vec{P}\cdot\hat{n} & \text{on the surface} \end{cases}$$

We also introduce "free charge" ρ_f for rest of charges (not result of polarization)

Within dielectrics, the total charge

$$\rho = \rho_b + \rho_f$$

and Gauss's law reads

$$\epsilon_0 \underbrace{\vec{\nabla}\cdot\vec{E}}_{\text{total field}} = \rho = \rho_b + \rho_f = -\vec{\nabla}\cdot\vec{P} + \rho_f$$

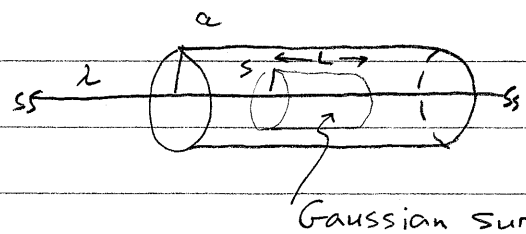
$\vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f$ We define $\underbrace{\vec{D}}_{\text{electric displacement}} \equiv \epsilon_0\vec{E} + \vec{P}$

So $\vec{\nabla}\cdot\vec{D} = \rho_f$ or $\oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}$

i) $\vec{D}$ has same dim. as $\vec{P}$ (= dipole/volume)

ii) $\vec{D}$ is "field" due to free charge only in matter.

Ex 4.4) A long straight wire w/ line charged density λ surrounded by rubber insulation of radius a



$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{D} = \vec{\rho}_f \\ \oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}\end{array}\right\} \rightarrow$ $D \cdot 2\pi s L = \lambda L$ $\quad \vec{D} = \dfrac{\lambda}{2\pi s}\hat{s}$

true anywhere (except surface)

but, $\vec{E} = \dfrac{1}{\epsilon_0}\vec{D} = \dfrac{\lambda}{2\pi\epsilon_0 s}\hat{s}$ for $s > a$ only ($\because \vec{P}$ not given)

· More about $\vec{D}$ No Coulomb's law for $\vec{D}$!

Even if $\vec{\nabla}\cdot\vec{D} = \rho_f$ $\rightarrow$ $\vec{D} \neq \frac{1}{4\pi}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho_f(\vec{r}')d\tau'$

($\because$ $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$

$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P} \neq 0$ in general)

· Boundary Conditions

The previous case of boundary conditions

: $\vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below}$

$\vec{E}^{\perp}_{above} - \vec{E}^{\perp}_{below} = \frac{1}{\epsilon_0}\sigma$

Now, for $\vec{D}$, from $\oint \vec{D}\cdot d\vec{a} = Q_{fenc}$

Therefore, we can say that for perpendicular component

$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$

while $\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$ gives us

$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$

free + (" induced)

total field

· Linear Dielectrics

· The polarization is (usually) proportional to the $\vec{E}$ so,

$\vec{P} \propto \vec{E}$

$= \epsilon_0\chi_e\vec{E}$ (why ϵ_0? To make χ_e dimensionless

$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$)

✓ χ_e : electric susceptibility (민감성)

✓ material follows $\vec{P} \propto \vec{E}$: linear dielectrics

✓ Since $\vec{E}$ is total field, one cannot compute $\vec{P}$ from

$\vec{P} = \epsilon_0\chi_e\vec{E}$ directly!

In linear media, we have

$\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} = \epsilon_0(1+\chi)\vec{E}$

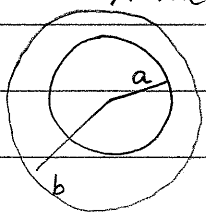
$\therefore$ $\vec{D} \propto \vec{E}$ and $\vec{D} = \epsilon\vec{E}$, $\epsilon \equiv \epsilon_0(1+\chi)$

↑ permittivity of the metal

Relative permittivity $\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$

(or dielectric constant) $\boxed{\text{Table : P187}}$

Ex4.5) A metal sphere $(r=a)$ with charge Q is surrounded by linear dielectric material of permittivity ϵ, radius b.



Q: potential at the center (w.r.t ∞)?

Starting w/ $\oint \vec{D} \cdot d\vec{a} = Q_{f\,enc}$, we get

$$\vec{D} = \frac{Q}{4\pi r^2}\hat{r} \quad \text{for} \quad r > a$$

Now, $\vec{D} = \epsilon \vec{E}$, we get

$$\vec{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{r} & a < r < b \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{r} & r > b \end{cases}$$

들 헷갈림!?

$\therefore$ potential at the center is (This is my way...)

$$V_f - V_i = \overset{\text{"}0}{V(\infty)} - V(0)$$

$$= -\int_0^\infty \vec{E} \cdot d\vec{\ell}$$

$$= -\int_a^b \frac{Q}{4\pi\epsilon}\frac{1}{r^2}\,dr - \int_b^\infty \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\,dr$$

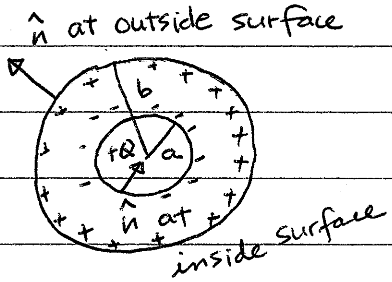
$$-V(0) = +\frac{Q}{4\pi\epsilon}\left(\frac{1}{b} - \frac{1}{a}\right) + \frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{b}\right)$$

$$\therefore V(0) = \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 b} + \frac{1}{\epsilon a} - \frac{1}{\epsilon b}\right) \text{ is obtained}$$

Q: Polarization in the dielectric?

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \frac{\epsilon_0 \chi_e Q}{4\pi\epsilon r^2}\hat{r}$$

So $\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$ (no bound charge inside) while $\sigma_b = \vec{P}\cdot\hat{n}$ gives



at outside surface: $\sigma_b = \dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon b^2}$

at inside surface: $\sigma_b = -\dfrac{\epsilon_0 \chi_e Q}{4\pi\epsilon a^2}$

4.4.2 Boundary Value Problems w/ Linear Dielectrics

From $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$$= -\vec{\nabla}\cdot\left(\epsilon_0 \chi_e \frac{\vec{D}}{\epsilon}\right) = -\frac{\chi_e}{1+\chi_e}\vec{\nabla}\cdot\vec{D} = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

✓ If free charges are not imbeded in the material ($\rho_f = 0$), then $\rho_b = 0$ → Charges are at the surface

Within material potential obeys Laplace's eq.

If we rewrite boundary conditions w/ free charge only:

$D^{\perp}_{above} - D^{\perp}_{above} = \sigma_f$

$(\vec{D} = \epsilon\vec{E})$ $\longrightarrow$ $\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f$

or, in terms of potential

$$\epsilon_{above}\left(-\frac{\partial V_{above}}{\partial n}\right) - \epsilon_{below}\left(-\frac{\partial V_{below}}{\partial n}\right) = \sigma_f$$

but of course the potential must be continuous:

$V_{above} = V_{below}$

Ex 4.7 A sphere of linear dielectric is placed in uniform $\vec{E}_0$

Q: $\vec{E}$ inside the sphere in terms of $\vec{E}_0$ and ϵ_r.

Our problem is to solve Laplace's eq for $V_{in}(r,\theta)$ when $r \le R$

$V_{out}(r,\theta)$ " $r \geq R$

With boundary conditions on potentials:

(i) $V_{in} = V_{out}$ at $r = R$

ii) Since $\rho_f = 0$ (no free charge)

$$\epsilon_{above} \frac{\partial V_{above}}{\partial r} - \epsilon_{below} \frac{\partial V_{below}}{\partial r} = \rho_f$$ gives us

$$\epsilon \frac{\partial V_{in}}{\partial r} = \epsilon_0 \frac{\partial V_{out}}{\partial r} \quad \text{at} \quad r = R$$

iii) Boundary condition at infinity

$$V_{out} \to -E_0 z = -E_0 r \cos\theta \quad \text{for} \quad r \gg R$$

We can assume that inside the Sphere

$$V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

$$V_{out}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta).$$

Applying the boundary condition (i):

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 R \underbrace{\cos\theta}_{P_1(\cos\theta)} + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta)$$

$$\left(\begin{array}{ll} \text{when } \ell=1: & A_1 R = -E_0 R + \dfrac{B_1}{R^2} \\ \text{for } \ell \neq 1: & A_\ell R^\ell = \dfrac{B_\ell}{R^{\ell+1}} \end{array} \right)$$ must be satisfied.

$\epsilon_r = \epsilon/\epsilon_0$

Boundary condition (ii):

$$\left.\frac{\partial V_{in}}{\partial r}\right|_{r=R} = \sum_{\ell=0} \ell A_\ell R^{\ell-1} P_\ell(\cos\theta)$$

$$\left.\frac{\partial V_{out}}{\partial r}\right|_{r=R} = -E_0\cos\theta - \sum_{\ell=} \frac{(\ell+1)}{R^{\ell+2}} B_\ell P_\ell(\cos\theta)$$

$$\left. \begin{array}{ll} \therefore \ \ell=1 & \epsilon_r A_1 = -E_0 - \dfrac{2B_1}{R^3} \\ \ell \neq 1 & \epsilon_r \ell A_\ell R^{\ell-1} = -\dfrac{(\ell+1)B_\ell}{R^{\ell+2}} \end{array} \right\}$$ must be satisfied.

For $\ell \neq 1$: $A_\ell R^\ell = \frac{B_\ell}{R^{\ell+1}} \rightarrow B_\ell = A_\ell R^{2\ell+1}$

$$\epsilon_r \ell A_\ell R^{\ell-1} = \frac{-(\ell+1)B_\ell}{R^{\ell+2}} = \frac{-(\ell+1)}{R^{\ell+2}} A_\ell R^{2\ell+1}$$

$$\rightarrow A_\ell \left[\epsilon_r \ell R^{\ell-1} + \frac{(\ell+1)}{1} R^{\ell-1} \right] = 0 \quad \forall \ell \neq 1$$

$\therefore A_\ell = 0$ so $B_\ell = 0$ for $\forall \ell \neq 1$.

For $\ell = 1$:

$$\begin{cases} A_1 R = -E_0 R + \frac{B_1}{R^2} \\ \epsilon_r A_1 = -E_0 - \frac{2B_1}{R^3} \end{cases}$$

$$\longrightarrow A_1 R = -E_0 R + \frac{R}{2}\left(-\epsilon_r A_1 - E_0\right)$$

$$R\left(1 + \frac{\epsilon_r}{2}\right) A_1 = -\frac{3}{2} E_0 R, \quad A_1 = \frac{2}{(2+\epsilon_r)} \frac{1}{R} \left(-\frac{3}{2}\right) E_0 R$$

$$\therefore A_1 = -\frac{3}{2+\epsilon_r} E_0$$

$$B_1 = A_1 R^3 + E_0 R^3 = -\frac{3}{2+\epsilon_r} E_0 R^3 + E_0 R^3$$

$$= \frac{\epsilon_r - 1}{\epsilon_r + 2} E_0 R^3$$

So, $V_{in}(r,\theta) = A_1 r^1 P_1(\cos\theta) = -\frac{3}{2+\epsilon_r} E_0 r \cos\theta = -\frac{3E_0}{2+\epsilon_r} z$

$$\vec{E}_{in} = \frac{3\vec{E}_0}{2+\epsilon_r}$$

In vacuum, $\epsilon \rightarrow \epsilon_0$, $\epsilon_r \rightarrow 1$, $\vec{E}_{in} = \vec{E}_0$!

4.4.3 Energy in Dielectric Systems

- To charge up, one has to work : $W = \frac{1}{2} C V^2$
- If the capacitor is filled with linear dielectric

일반물리 $\kappa \rightarrow \epsilon_r$

$$C = \epsilon_r C_{vac}$$

and the capacitance is increased. Work is increased

($\because$ more free charge to store to achieve a given potential since $\vec{E}$ is canceled by bound charges)

In vacuum, the energy stored in any electrostatic system

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau.$$

How is it going to be changed in the presence of linear dielectrics

$$W_0 = \frac{1}{2}CV^2 \xrightarrow{C = \epsilon_r C_{vac}} W = \epsilon_r \frac{1}{2}CV^2$$ have been seen.

So, $W = \frac{\epsilon_0}{2}\int E^2 d\tau \times \epsilon_r = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$?

- Suppose the dielectric material is fixed in position.
- We bring in ρ_f (free charge) $\rightarrow$ polarization will change but we are interested in the work done on the incremental free charge:

$$\Delta W = \int (\Delta\rho_f) V \, d\tau$$

V ← electric potential

Sin $\vec{\nabla}\cdot\vec{D} = \rho_f$, $\Delta\rho_f = \vec{\nabla}\cdot(\Delta\vec{D})$, we have

$$\Delta W = \int [\vec{\nabla}\cdot(\Delta\vec{D})] V d\tau$$

$$= \int \vec{\nabla}\cdot[\Delta\vec{D} V] d\tau - \int \Delta\vec{D}\cdot\overbrace{(\vec{\nabla}V)}^{-\vec{E}} d\tau$$

$$= \underbrace{\oint_S (\Delta\vec{D} V)\cdot d\vec{a}}_{0} + \int (\Delta\vec{D})\cdot\vec{E}\, d\tau$$

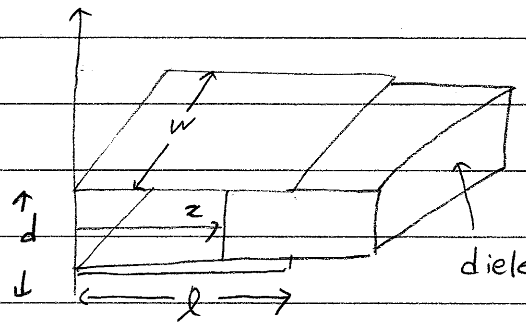
(see page 94. This surface can be arbitrarily large.)

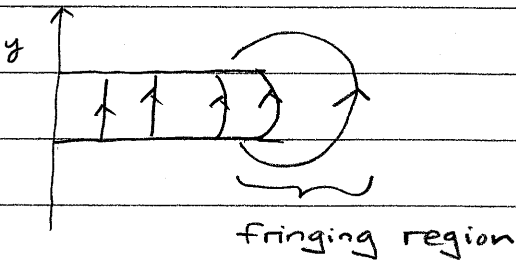
$$\Delta W = \int (\Delta\vec{D})\cdot\vec{E}\, d\tau$$

Now, $\frac{1}{2}\Delta(\vec{D}\cdot\vec{E}) = \frac{1}{2}\Delta(\epsilon E^2) = \epsilon(\Delta E)E = (\Delta\vec{D})\cdot\vec{E}$

$$\rightarrow \Delta W = \Delta\left(\frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau\right) \Rightarrow W = \frac{1}{2}\int \vec{D}\cdot\vec{E}\, d\tau$$

4.4.4 Forces on Dielectrics





Force acting on the dielectrics?

W : energy of the system (may be $\propto x$)

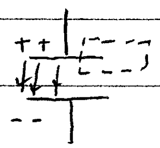
If one pulls out it by dx, the energy chage is equal to the work done:

$$dW = F_{me}\, dx \qquad F_{me} = -F \leftarrow \text{electrostatic force on the dielectric.}$$

$$\therefore F = -\frac{dW}{dx}$$

- The energy stored in the capacitor : $W = \frac{1}{2} C V^2$
- What is C ?

In vacuum,

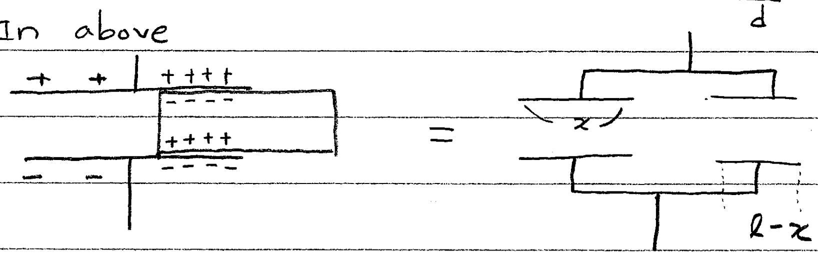


Gausss law gives

$$\frac{\sigma A}{\epsilon_0} = E \cdot A \qquad \underbrace{\frac{\overbrace{\sigma A}^{Q} d}{\epsilon_0}} = \overbrace{E d}^{V} A$$

$$\rightarrow Q = \frac{\epsilon_0 A}{d} V \quad \therefore C_{vac} = \frac{\epsilon_0 A}{d}$$

In above



$C = \epsilon_r C_{vac}$

$$\therefore C_{left} = \frac{\epsilon_0}{d} \cdot W \cdot x \qquad C_{right} = \epsilon_r \cdot \frac{\epsilon_0}{d} W (l - x)$$

$$C = C_{left} + C_{right}$$

$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$

$$= \frac{\epsilon_0 W}{d}\left(x + \epsilon_r (l - x) \right) = \frac{\epsilon_0 W}{d}\left[\epsilon_r l - (\epsilon_r - 1) x \right]$$

$$= \frac{\epsilon_0 W}{d}\left[\epsilon_r l - \chi_e x \right] \quad \text{is obtained}$$

From $W = \frac{1}{2}\frac{Q^2}{C}$

$$F = -dW/dx = \frac{1}{2}Q^2 \frac{1}{C^2} \cdot \frac{dC}{dx} = \frac{1}{2}V^2 \frac{dC}{dx}$$

$$\frac{dC}{dx} = \frac{\epsilon_0 W}{d}(-\chi_e)$$

$$\therefore \quad F = -\frac{\epsilon_0 \chi_e W}{2d} V^2$$

(−): dielectric is pulled into the capacitor