Potentials

3.1 Laplace's equation

One of main tasks is to find $\vec{E}$ for given $\rho(\vec{r})$

(Not $\rho(\vec{r},t)$ now.)

Formally, $\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}')\,d\tau'$

but the integration is usually complicated.

one can try $V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}^2}\rho(\vec{r}')\,d\tau'$ ①

and get $\vec{E} = -\vec{\nabla} V$ but again this integral may not be solvable.

The differential form we had was (Poission eq.)

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

and is mathematically equivalent to ① w/ appropriate boundary conditions.

If we are interested in $V(\vec{r})$ at a space point w/o charge at $\vec{r}$,

$\nabla^2 V = 0$ (of cause we may have charge elsewhere)

or $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

- Laplace eq. in one dimension

$\frac{d^2 V}{dx^2} = 0 \rightarrow V = mx + b$ (general sol.)

: straight line.

Note: m and b are to be fixed by boundary conditions

Ex) $V(x=1) = 4$, $V(x=5) = 0$

$4 = m + b$ $\quad 0 = 5m + b \rightarrow m = -1$, $b = 5$,

$V(x) = -x + 5$

Things to note (too obvious?)

1. $V(x) = \frac{1}{2}[V(x+a) + V(x-a)]$

$(\because \frac{1}{2}[m(x+a)+b + m(x-a)+b] = mx+b = V(x))$

$\therefore$ $V(x)$ is the average of $V(x+a)$ and $V(x-a)$.

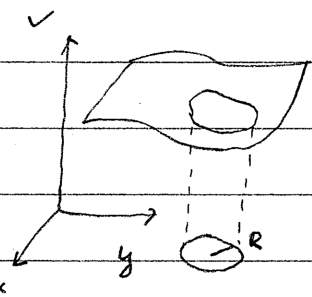
2. No local maxima or minima

- Laplace's eq. in two dimensions.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \qquad \text{(not ordinary diff. eq. any more)}$$

: no obvious sol. exists in this case.

1. $V(x,y) = \frac{1}{2\pi R}\oint_{\text{circle}} V\, dl$

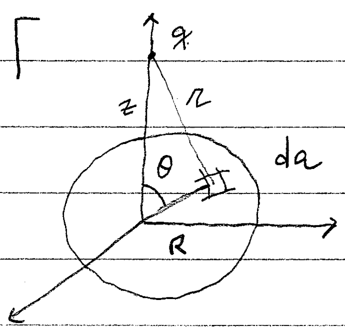


2. V has no local maxima or minima. Extreme values of V must occur at the boundaries

- Laplace's eq. in 3 dimensions

1. $V(\vec{r}) = \frac{1}{4\pi R^2}\oint_{\text{sphere}} V\, da$

2. V can have no local maxima or minima.



The potential at a point on the surface is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

where $r^2 = z^2 + R^2 - 2zR\cos\theta$

$(\vec{r} = \vec{z} - \vec{R})$

$$V_{ave} = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da \qquad\qquad -\sin\theta \, d\theta = d(\cos\theta)$$

$$= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int \left[z^2 + R^2 - 2zR\cos\theta \right]^{-1/2} R^2 \sin\theta \, d\theta \, d\phi$$

$\theta : [0, \pi]$

$$= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} R^2 \cdot 2\pi \int_{-1}^{1} \left[z^2 + R^2 - 2zR\cos\theta \right]^{-1/2} d(\cos\theta)$$

$\cos\theta : [1, -1]$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{2} \int_{-1}^{1} \left(z^2 + R^2 - 2zRx \right)^{-1/2} dx \qquad\qquad \int x^p \, dx = \frac{1}{p+1} x^{p+1}$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{2} \left(\frac{1}{-\frac{1}{2}+1} \right) \left(z^2 + R^2 - 2zRx \right)^{1/2} \cdot \frac{1}{(-2zR)} \Bigg|_{-1}^{1}$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{2zR} \left(z^2 + R^2 - 2zRx \right)^{1/2} \Bigg|_{1}^{-1}$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{2zR} \left[z + R - (z - R) \right] = \frac{q}{4\pi\epsilon_0} \frac{1}{z}.$$

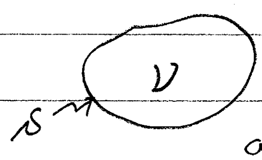
$\therefore \; V_{ave} = V(z)!$

Boundary conditions and Uniqueness Theorems

→ Laplace's eq. does not by itself determine V.

In addition, suitable boundary conditions must be supplied.

First uniqueness theorem: The solution to Laplace's eq. in some volume $\mathcal{V}$ is uniquely determined if potential V is specified on the boundary surface $\mathcal{S}$.



Suppose two sol. to Laplace eq. $\nabla^2 V_1 = 0$, $\nabla^2 V_2 = 0$ and both of which assume the specified value on the surface.

we define $V_3 \equiv V_1 - V_2$ then $\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$ and $V_3 (\text{on } S) = 0$.

→ From "no local maxima/minima, $V_3 = 0$ everywhere.

$\therefore \; V_1 = V_2$

Conductors and the 2nd Uniqueness Theorem

2nd uniqueness theorem :

In a volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density ρ, $\vec{E}$ is uniquely determined if the total charge on each conductor is given.

[Suppose two fields $\vec{E}_1$ and $\vec{E}_2$ satisfy the conditions of the problem.

$$\vec{\nabla}\cdot\vec{E}_1 = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla}\cdot\vec{E}_2 = \frac{1}{\epsilon_0}\rho$$

Apply Gauss's law w/ a Gaussian surface enclosing each conductor :

$$\oint \vec{E}_1\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_i \qquad \oint \vec{E}_2\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_i$$

Likewise, for the outer boundary

$$\oint \vec{E}_1\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{tot} \qquad \oint \vec{E}_2\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{tot}$$

Define $\vec{E}_3 \equiv \vec{E}_1 - \vec{E}_2$, $\left\{ \begin{array}{ll} \vec{\nabla}\cdot\vec{E}_3 = 0 & \text{in the region between conductors} \\ \oint \vec{E}_3\cdot d\vec{a} = 0 & \text{over each boundary surface.} \end{array} \right.$

Now, $\vec{\nabla}\cdot(V_3\vec{E}_3) = V_3 \underbrace{\vec{\nabla}\cdot\vec{E}_3}_{=0} + \vec{E}_3\cdot(\vec{\nabla}V_3) = -(E_3)^2$

If we use divergence theorem w/ integrating over $\mathcal{V}$

$$\int_{\mathcal{V}} \vec{\nabla}\cdot(V_3\vec{E}_3)\,d\tau = \oint V_3\vec{E}_3\cdot d\vec{a} = -\int_{\mathcal{V}} (E_3)^2\,d\tau$$

$$= V_3 \underbrace{\oint \vec{E}_3\cdot d\vec{a}}_{=0}$$ on the surface (V_3 constant)

$E_3 = 0$, $\vec{E}_1 = \vec{E}_2$!]

- The method of Images

Q: q is held a distance d above an infinite grounded conducting plane. $V = ?$

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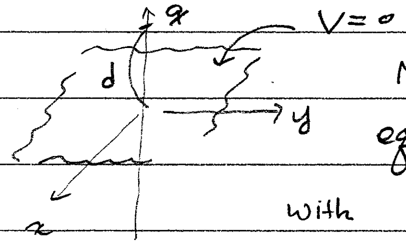


Mathematically, it is solving Poisson's eq. in $z>0$ with q at $(0,0,d)$

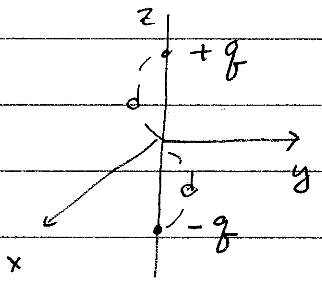
with

1) $V=0$ when $z=0$

2) $V \to 0$ when $x^2+y^2+z^2 \gg d^2$

Trick:

Let's imagine a situation of



For this configuration,

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

and this satisfy above two boundary conditions $\longrightarrow$ $V(x,y,z)$ should be the potential for the original problem ($\because$ uniqueness theorem)

- Induced Surface Charge.

We know that $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$

In this case, $\hat{n} = \hat{z}$. So $\sigma = -\epsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$

$$\left.\frac{\partial V}{\partial z}\right|_{z=0} = \frac{1}{4\pi\epsilon_0}\left[\frac{-q\,2(z-d)}{2(x^2+y^2+(z-d)^2)^{3/2}} + \frac{q\,2(z+d)}{2(x^2+y^2+(z+d)^2)^{3/2}}\right]\Bigg|_{z=0}$$

$$= \frac{1}{4\pi\epsilon_0}\left[\frac{qd}{(x^2+y^2+d^2)^{3/2}}\cdot 2\right] \qquad \therefore \sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

What about total charge?

$Q = -q$ expect

$$Q = \int \sigma\, da = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi (r^2+d^2)^{3/2}}\, r\, dr\, d\phi$$

$$= (-qd) \int_0^{\infty} (r^2+d^2)^{-3/2}\, r\, dr \qquad \left(\sqrt{r^2+d^2}\right)' = -\frac{r}{(r^2+d^2)^3}$$

$$= (qd) \cdot \frac{1}{\sqrt{r^2+d^2}} \Big|_0^{\infty} = -q$$

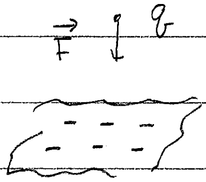
$$\text{or } (-qd)\int_0^{\infty} (r^2+d^2)^{-3/2} r\, dr = (-qd) \int_0^{\pi/2} \left[d(1+\tan^2\theta)\right]^{-3/2} d\tan\theta\, d\sec^2\theta\, d\theta$$

$r = d\tan\theta \qquad dr = d\sec^2\theta\, d\theta$

$$= (-qd)\, d^{-3+2} \int_0^{\pi/2} \sec^{-3}\theta\, \tan\theta\, \sec^2\theta\, d\theta$$

$$= (-q) \int_0^{\pi/2} \cos\theta \cdot \frac{\sin\theta}{\cos\theta}\, d\theta = (-q)(-\cos\theta)\Big|_0^{\pi/2} = -q$$

- Force and Energy

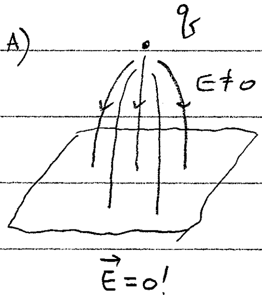


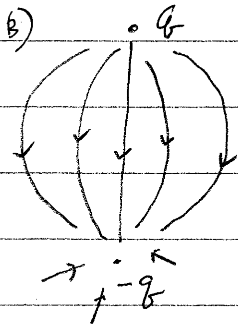
The charge q is attracted toward the plane. Since the potential is same for $+q$ and $-q$ at the distance of $(2d)$

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

V same $\rightarrow$ E same $\downarrow$ F same

Energy?





We can guess that the right one has twice $\vec{E}$ field so energy will be twice!

B) The work to assemble two charges

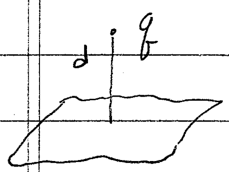
$$W = \frac{1}{4\pi\epsilon_0}\frac{q\cdot(-q)}{2d} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}$$

Work required to bring q in from infinity. Force is $-\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}\hat{z}$

A) $W = \int_{\infty}^{d} \vec{F}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}\,dz = \frac{q^2}{4\pi\epsilon_0}\cdot\frac{1}{4}\left(-\frac{1}{z}\right)\Big|_{\infty}^{d}$

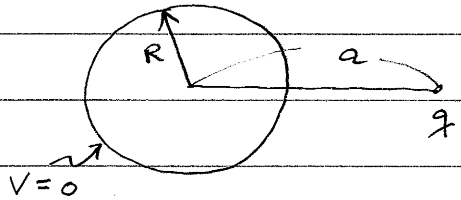
$d\vec{l}\downarrow$

$= -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d}$



· Other Image Problems

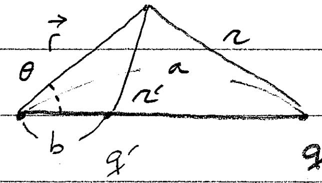
Ex3.2 A point charge q is situated a distance a from the center of a grounded conducting sphere (radius R). Potential outside the sphere?



Consider the following situation:

point charge q, another charge $q' = -\frac{R}{a}q$

placing a distance $b = \frac{R^2}{a}$



The potential of this conf. is

$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$

Now, $V(R) = ?$ Cosine law says

$\mathscr{r}^2 = R^2 + a^2 - 2Ra\cos\theta$

$\mathscr{r}'^2 = R^2 + b^2 - 2Rb\cos\theta = R^2 + \frac{R^4}{a^2} - 2R\frac{R^2}{a}\cos\theta$

$\frac{a^2}{R^2}\cdot\mathscr{r}'^2 = a^2 + R^2 - 2Ra\cos\theta = \mathscr{r}^2$

$\therefore\ \mathscr{r}' = \frac{R}{a}\mathscr{r}$

so, $V(R) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \left(-\frac{R}{a}\right)q\cdot\frac{1}{\frac{R}{a}\mathscr{r}}\right) = 0.$

$\therefore\ V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$ must be the solution of the problem

Force between the charge and the Sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0}\frac{q\cdot\left(-\frac{R}{a}\right)q}{\left(a-\frac{R^2}{a}\right)^2}\frac{\times a^2}{\times a^2}$$

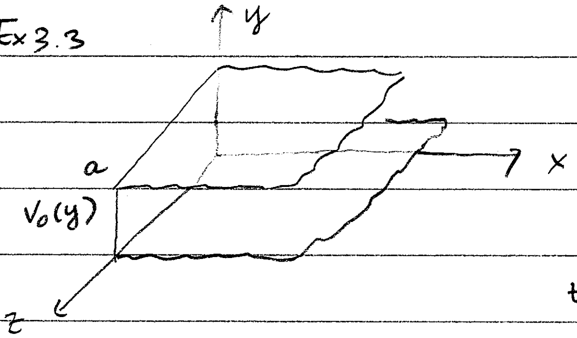
$$= -\frac{1}{4\pi\epsilon_0}\frac{q^2 R a}{(a^2-R^2)^2}$$

3.3 Separation of Variables

Solving Laplace's eq. directly. $\rightarrow$ Using separation of Variables

3.3.1 Cartesian coord.

Ex 3.3



Two infinite grounded metal plate: one at $y=0$, the other at $y=a$. The left end, at $x=0$, is closed off w/ an infinite strip, insulated from two plates, at a specific potential $V_0(y)$. Potential inside this slot?

This configuration is independent of z. $\rightarrow$ 2-dim. problem.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

w/ boundary conditions
- i) $V=0$ when $y=0$
- ii) $V=0$ when $y=a$
- iii) $V=V_0(y)$ " $x=0$
- iv) $V\rightarrow 0$ as $x\rightarrow\infty$

Let's assume $V(x,y) = X(x)\cdot Y(y)$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0 \rightarrow \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

This is a form of $f(x) + g(y) = 0$ for all x and y.

$\rightarrow$ $f(x) = C_1$ and $g(y) = C_2$, $C_1 + C_2 = 0$!

Let's assume $C_1 > 0$ (this choice will be clearer later)

$$\rightarrow \quad \frac{d^2X}{dx^2} = k^2 X \qquad \frac{d^2Y}{dy^2} = -k^2 Y$$

$$X(x) = Ae^{kx} + Be^{-kx} \qquad Y(y) = C\sin ky + D\cos ky$$

so, $V(x,y) = (Ae^{kx} + Be^{-kx}) \cdot (C\sin ky + D\cos ky)$

iv) $V \to 0$ as $x \to \infty$ force us $A = 0$.

$\therefore V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$

i) $V = 0$ when $y = 0$ force us $D = 0$

$\therefore V(x,y) = Ce^{-kx}\sin ky$ $\quad$ $\ulcorner n=0$ not interesting.

ii) $V = 0$ when $y = a$ $\quad$ " $\quad ka = n\pi \quad (n = 1, 2, 3 \ldots)$

Now, it is obvious that $C_1 < 0$ will not satisfy iv) at all.

So, separation of variables gives us in infinite family of solutions (one for each n).

Our general sol. becomes

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

Note: we have not used boundary cond iii) yet!

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y)$$

↳ This requirement will decide all C_n uniquely? (must be!?)

$$\sum_{n=1}^{\infty} C_n \underbrace{\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right)dy}_{\substack{=0 \text{ if } n \neq n' \\ = \frac{a}{2} \text{ if } n = n'}} = \int_0^a V_0(y)\sin\left(\frac{n'\pi}{a}y\right)dy$$

$\left(\int_0^a \sin^2\left(\frac{n\pi}{a}y\right)dy = \frac{a}{2} \qquad \int_0^a \left(\sin^2\left(\frac{n\pi}{a}y\right) + \cos^2\left(\frac{n\pi}{a}y\right)\right)dy = a \right.$

$$\therefore C_n = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi}{a}y\right)dy$$

so the solution is completely determined on given $V_0(y)$.

Suppose we think of $V_0(y) = V_0 = $ constant

Then $c_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi}{a}y\right)dy = \frac{2V_0}{a}\cdot\left(\frac{-a}{n\pi}\right)\cos\left(\frac{n\pi}{a}y\right)\Big|_0^a$

$$= \frac{2V_0}{n\pi}\left(1-\cos n\pi\right) = \begin{cases} 0 & n: \text{even} \\ \frac{4V_0}{n\pi} & n: \text{odd}\end{cases}$$

$$\therefore\ V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\cdots}\frac{1}{n}e^{-\frac{n\pi}{a}x}\sin\left(\frac{n\pi}{a}y\right)$$

This can be written as a closed form: Using $\sin\theta = \operatorname{Im}(e^{i\theta})$

$$V(x,y) = \frac{4V_0}{\pi}\operatorname{Im}\sum_{n=1,3,5\ldots}\frac{1}{n}e^{\left(\frac{in\pi}{a}\right)(ix+y)}$$

$z \equiv e^{\frac{i\pi}{a}(y+ix)}$ gives us

Not so important ?!

$$V(x,y) = \frac{4V_0}{\pi}\operatorname{Im}\sum_{n=1,3,5\ldots}\frac{z^n}{n}$$ Using $\ln(1+z) = z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \ldots$

$$\sum_{n=1,3,5\ldots}\frac{z^n}{n} = \frac{1}{2}\left[\ln(1+z) - \ln(1-z)\right] = \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)$$

$$\therefore\ V(x,y) = \frac{2V_0}{\pi}\operatorname{Im}\left[\ln\frac{(1+z)}{(1-z)}\right]$$

$$\frac{1+z}{1-z} = \frac{(1+z)(1-z^*)}{|1-z|^2} = \frac{1-|z|^2+2i\operatorname{Im}z}{|1-z|^2}$$

Now, in general, $\operatorname{Im}[\ln(re^{i\theta})] = \theta = \tan^{-1}\frac{\operatorname{Im}}{\operatorname{Re}}$

So, $\operatorname{Im}\left[\ln\left(\frac{1+z}{1-z}\right)\right] = \tan^{-1}\left(\frac{2\operatorname{Im}z}{1-|z|^2}\right)$ is satisfied.

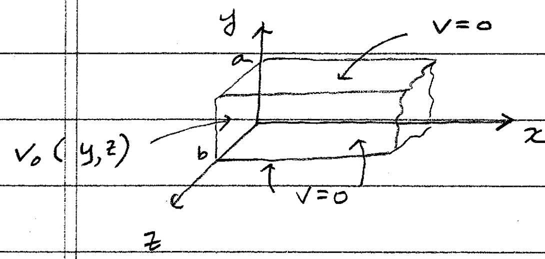
$$2\operatorname{Im}z = 2\operatorname{Im}e^{\frac{i\pi}{a}(y+ix)} = 2\cdot\sin\left(\frac{\pi}{a}y\right)e^{-\frac{\pi}{a}x}$$

$$1-|z|^2 = 1-e^{-\frac{2\pi}{a}x}$$

$$\therefore\ \frac{2\operatorname{Im}z}{1-|z|^2} = \frac{2\sin\left(\frac{\pi}{a}y\right)e^{-\frac{\pi}{a}x}}{1-e^{-\frac{2\pi}{a}x}} = \frac{2\sin\left(\frac{\pi}{a}y\right)}{e^{\pi/a\,x}-e^{-\pi/a\,x}}$$

$$= \frac{\sin\left(\frac{\pi y}{a}\right)}{\sinh\left(\frac{\pi x}{a}\right)}\qquad \therefore\ V(x,y) = \frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin(\pi y/a)}{\sinh\left(\frac{\pi x}{a}\right)}\right)$$

Ex 3.5 An infinitely long rectangular metal pipe. At $x=0$, $V_0(y,z)$ other metal pipe plane all grounded. Potential inside the pipe?



Now, this is a 3-dim. Problem.

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

w/ boundary conditions

i) $V=0$ when $y=0$
ii) $V=0$ " $y=a$
iii) $V=0$ " $z=0$
iv) $V=0$ " $z=b$
v) $V\to\infty$ " $x\to\infty$
vi) $V=V_0(y,z)$ " $x=0$

Let's begin with

$$V(x,y,z)=X(x)\,Y(y)\,Z(z)$$

then, the above Laplace's eq becomes

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=C_1 \qquad \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2 \qquad \frac{1}{Z}\frac{d^2Z}{dz^2}=C_3, \qquad C_1+C_2+C_3=0$$

We choose $C_1>0$, $C_2<0$ $C_3<0$. (why?)

Let's set $C_2=-k^2$, $C_3=-l^2$ and $C_1=k^2+l^2$. 3 diff. eqns becomes

$$\frac{d^2X}{dx^2}=(k^2+l^2)X \qquad \frac{d^2Y}{dy^2}=-k^2Y \qquad \frac{d^2Z}{dz^2}=-l^2Z$$

The solutions are

$$X(x)=Ae^{\sqrt{k^2+l^2}\,x}+Be^{-\sqrt{k^2+l^2}\,x}$$
$$Y(y)=C\sin ky+D\cos ky$$
$$Z(z)=E\sin lz+F\cos lz$$

v): $A=0$ i) $D=0$ iii) $F=0$

ii) $ka=n\pi$ iv) $lb=m\pi$: n, m are positive integers

So, $V(x,y,z)=C\,e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$

The most general sol. is the linear comb. of all possible solutions:

$$V(x,y,z)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}C_{n,m}\,e^{-\pi\sqrt{(n/a)^2+(m/b)^2}\,x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right).$$

To obtain $C_{n,m}$, $\times \sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{m'\pi}{b}z\right)$ and integrate on $V(0,y,z) = V_0(y,z)$ (vi)

$$\int_0^a\int_0^b V_0(y,z)\sin\left(\frac{n'\pi}{a}y\right)\sin\left(\frac{m'\pi}{b}z\right)dy\,dz$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{n'\pi}{a}y\right)dy\int_0^b \sin\left(\frac{m\pi}{b}z\right)\sin\left(\frac{m'\pi}{b}z\right)dz$$

$$\therefore\ C_{n,m} = \frac{4}{ab}\int_0^a\int_0^b V_0(y,z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)dy\,dz$$

Now, if $V_0(y,z) = V_0$ (complete constant)

$$C_{n,m} = \frac{4V_0}{ab}\int_0^a \sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)dy\,dz$$

$$= \frac{16V_0}{\pi^2 nm}.$$

$$\therefore\ V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n,m=1,3,5}^{\infty}\frac{1}{nm}e^{-\pi\sqrt{(n/a)^2+(m/b)^2}}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

- Spherical Coordinates

Laplace eq. in spherical coord.? (Can or should you remember this?)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Suppose our problem has azimuthal symmetry.

$\rightarrow V = V(r,\theta)$ not $V(r,\theta,\phi)$

So
$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0.$$

Again, separation of variables:

$$V(r,\theta) = R(r)\,\Theta(\theta)$$

$\frac{1}{V}\times$
$$\underbrace{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)}_{\text{depends only on } r} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)}_{\text{depends only on } \theta.} = 0$$

So each term must be a constant.

Let's take a look at the angular part first.

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+C\Theta=0$$

Mathetically, this is known as Legendre eq. when

$C=\ell(\ell+1)$ and $\ell=0,1,2,3,\ldots$

so $$\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\ell(\ell+1)\sin\theta\,\Theta=0$$

and sol. to this diff. eq. are Legendre polynomials

$$\Theta(\theta)=P_\ell(\cos\theta)$$

with the definition $P_\ell(x)\equiv\frac{1}{2^\ell\ell!}\left(\frac{d}{dz}\right)^\ell(x^2-1)^\ell$

$\therefore$ Radial part becomes

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\ell(\ell+1)\quad\text{or}\quad\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\ell(\ell+1)R$$

General Solution? Let's try

i) $R\propto r^\ell$.

$$\frac{d}{dr}\left(r^2\cdot\ell r^{\ell-1}\right)=\frac{d}{dr}\left(\ell r^{\ell+1}\right)=\ell(\ell+1)r^\ell=\ell(\ell+1)R!$$

ii) $R\propto r^{-(\ell+1)}$

$$\frac{d}{dr}\left(r^2(-)(\ell+1)r^{-(\ell+2)}\right)=-\frac{d}{dr}\left((\ell+1)r^{-\ell}\right)$$

$$=(\ell+1)\ell r^{-(\ell+1)}=\ell(\ell+1)R$$

So the general sol. becomes

$$R=Ar^\ell+\frac{B}{r^{\ell+1}}$$

$\therefore$ With azimuthal symmetry, the most general sol. to Laplace's eq. is

$$V(r,\theta)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

속이 빈

Ex 3.6 The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, radius R. Potential inside?

In this case, $B_\ell = 0 \ \forall \ell$ (otherwise V blows up @ origin)

$$\therefore \quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta)$$

At $r=R$, $\quad V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = V_0(\theta)$

Legendre polynomials have following orthonalities:

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\,dx = \int_0^\pi P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta\, d\theta$$

$x = \cos\theta$ $\qquad [-1, 1] \to [\pi, 0]$

$dx = -\sin\theta\, d\theta$

$$= \begin{cases} 0 & \text{if } \ell' \neq \ell \\ \dfrac{2}{2\ell+1} & \text{if } \ell' = \ell \end{cases}$$

Therefore

$$A_\ell R^\ell \frac{2}{2\ell+1} = \int_0^\pi V_0(\theta) P_\ell(\cos\theta) \sin\theta\, d\theta$$

and in principle this gives the all A_ℓ's $\longrightarrow$ sol. to Laplace's eq

For instance,

$$V_0(\theta) = k \sin^2\left(\frac{\theta}{2}\right), \quad k = \text{constant}$$

$\int_0^\pi k \sin^2\left(\frac{\theta}{2}\right) P_\ell(\cos\theta) \sin\theta\, d\theta$? How do we solve it?

The trick is : $P_0(x) = 1$, $\quad P_1(x) = x \qquad \sin^2\frac{\theta}{2} = 1 - \cos\theta$

$$\therefore \quad V_0(\theta) = \frac{k}{2}(1 - \cos\theta)$$

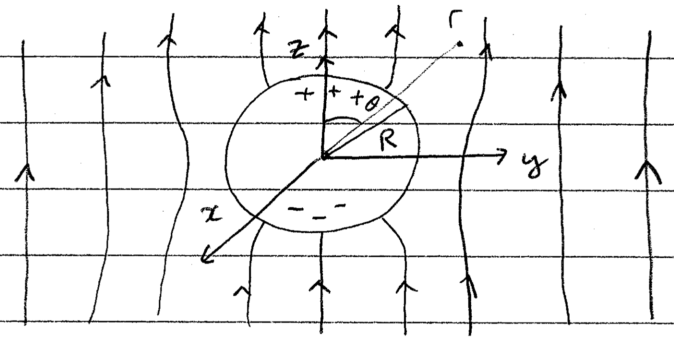
$$= \frac{k}{2}\left(P_0(\cos\theta) - P_1(\cos\theta)\right)$$

$$\therefore \quad A_0 = \frac{1}{2} \cdot \frac{k}{2} \underbrace{\int_0^\pi P_0^2(\cos\theta) \sin\theta\, d\theta}_{=2} = \frac{k}{2}$$

$$A_1 \underset{(-)}{=} \frac{2+1}{2R} \cdot \frac{k}{2} \underbrace{\int_0^\pi P_1^2(\cos\theta) \sin\theta\, d\theta}_{\frac{2}{2+1}} = \frac{-k}{2R}$$

$$\therefore V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R} P_1(\cos\theta)\right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

Ex 3.8 An uncharged metal sphere (radius = R) in uniform electric field $\vec{E} = E_o \hat{z}$. V at the outside sphere?



Note: the sphere is equipotential. Let's set V on the surface is zero $V(r=R)=0$

Far from sphere, the field is $E_o \hat{z}$, $\therefore V \to -E_o z + C$ at far from sphere.

Boundary conditions for this problem:

i) $V=0$ when $r=R$

ii) $V \to -E_o r\cos\theta$ for $r \gg R$ (we can always neglect constant (consistently

We can start from

$$V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

Applying i) gives

$$0 = \sum_{\ell=0}^{\infty}\left(A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}}\right)P_\ell(\cos\theta)$$

$$\therefore \quad A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} = 0 \quad \text{for all } \ell$$

$B_\ell = -A_\ell R^{2\ell+1}$ must be satisfied $\ni \forall \ell$

$$\hookrightarrow \quad V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell \left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}}\right)P_\ell(\cos\theta)$$

For $r \gg R$, $|r^\ell| \gg \left|\frac{R^{2\ell+1}}{r^{\ell+1}}\right|$. so ii) requires that

$$\sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) = -E_o r\cos\theta = -E_o r P_1(\cos\theta)$$

$\therefore$ $A_1 = -E_o$ and all other A_ℓ's zero!

$$V(r,\theta) = (-E_0)\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

→ 1st term: $-E_0 r\cos\theta$ nothing but from the external field

→ 2nd term: $E_0 \frac{R^3}{r^2}\cos\theta$: contribution from the induced charge.

Induced charge density?

$$\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R} = -\epsilon_0 (-E_0)\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta \Big|_{r=R}$$

$$= \epsilon_0 E_0 (3)\cos\theta \quad \therefore \quad \sigma > 0,\ \theta \in [0, \pi/2]$$

$$\sigma < 0 \quad \theta [\pi/2, \pi]$$

(*) Visualizing $\vec{E}$?

$$V(x,y,z) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

$$= -E_0\left(r - \frac{R^3}{r^2}\right)\frac{z}{r}$$

$$= -E_0 z + E_0 R^3 \frac{1}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial V}{\partial y}\Big|_{x=0} = E_0 R^3 \left(-\frac{3}{2}\right)(y^2+z^2)^{-5/2}\cdot 2y$$

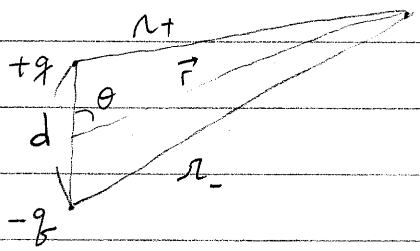
$$= -\epsilon_0 R^3\, 3zy\,(y^2+z^2)^{-5/2}$$

$$\frac{\partial V}{\partial z}\Big|_{x=0} = -E_0 + E_0 R^3 \left[\frac{(y^2+z^2)^{3/2} - z\cdot\frac{3}{2}\cdot(y^2+z^2)^{1/2}\cdot 2z}{(y^2+z^2)^3}\right]$$

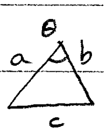
3.4 Multipole Expansion

· Approximate Potentials at large Distances

Ex: electric dipole



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$



$c^2 = a^2 + b^2 - 2ab\cos\theta$

$$r_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta$$

$$= r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

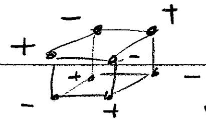
When $r \gg d$, $\frac{1}{r_\pm} \approx \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-1/2}$

$$\simeq \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

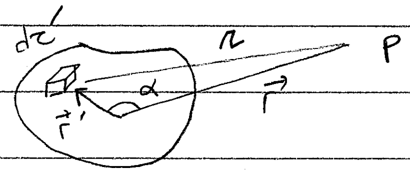
$\therefore \frac{1}{r_+} - \frac{1}{r_-} \cong \frac{d}{r^2}\cos\theta$ is obtained.

$\therefore\ V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0} \frac{q d \cos\alpha}{r^2}$: dipole potential.

(cf) + (monopole) $V \sim \frac{1}{r}$; dipole $V \sim \frac{1}{r^2}$; quadrupole $V \sim \frac{1}{r^3}$; Octopole $V \sim \frac{1}{r^4}$



· Now, Let's develop a systematic expansion



The potential at $\vec{r}$ is given by

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} \rho(\vec{r}\,') \, d\tau'$$

Using the law of cosines,

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

If we define $\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$, $\mathscr{r}$ becomes

$\mathscr{r} = r\sqrt{1+\epsilon}$. When P is well outside, $(r \gg 1)$

$\epsilon \ll 1$, and $\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \cdots\right)$

or,

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\alpha + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n \underbrace{P_n(\cos\alpha)}_{\text{Legendre polynomials}}$$

$$\therefore\ V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\,\rho(\vec{r}\,')\,d\tau'$$

- The Monopole and Dipole Terms.

When $n=0$,

$$V(r) \text{ has } \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \overset{1}{\overset{\parallel}{P_0}}\, \rho(\vec{r}')\, d\tau' = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

and is called the monopole term. This is zero if $\int \rho\, d\tau$;

The next term is $(n=1)$

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \underset{\cos\alpha}{\underset{\parallel}{P_1(\cos\alpha)}}\, \rho(\vec{r}')\, d\tau'$$

$$r'\cos\alpha = \hat{r}\cdot\vec{r}'$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot \underbrace{\int \vec{r}' \rho(\vec{r}')\, d\tau'}_{\equiv \vec{p} \text{ dipole moment}}$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

- $\vec{E}$ of a dipole

$$V_{dip}(\vec{r}) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

$$E_r = -\frac{\partial V}{\partial r} = (-)\frac{p\cos\theta}{4\pi\epsilon_0}\cdot(-2)r^{-3} = \frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

$$\therefore \vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$