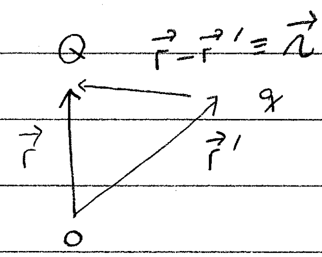
2. Electrostatics

- Coulomb's law : force on Q due to q ?

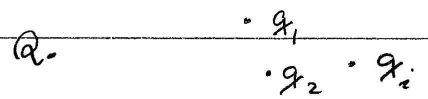
$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{qQ}{\mathscr{r}^2} \hat{\mathscr{r}}$$

ϵ_0 : permittivity of free space

$$\epsilon_0 = 8.85 \times 10^{-12} \ \frac{C^2}{N \cdot m^2}$$



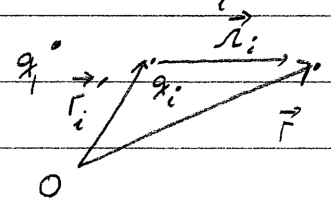
- Superposition principle



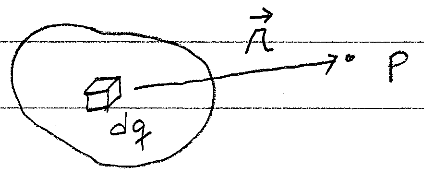
$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$$

- Electric field for several point charges

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{\mathscr{r}_i^2} \hat{\mathscr{r}}_i$$



- Continuous charge distribution



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}^2} \hat{\mathscr{r}} \, dq$$

$\swarrow$ charge densities

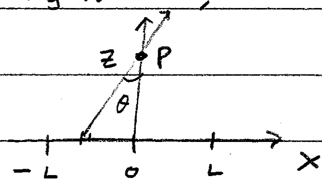
i) line charge : $dq \rightarrow \lambda \, dl'$

ii) surface " : $\rightarrow \sigma \, da'$

iii) volume " : $\rightarrow \rho \, d\tau'$

For example, $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0} \displaystyle\int \frac{\rho(\vec{r}\,')}{\mathscr{r}^2} \hat{\mathscr{r}} \, d\tau'$

Ex 2.2) Length $2L$, uniform line charge λ. E field at z above?



Symmetric w.r.t $x \to -x$ $\longrightarrow$ No x-component of E field!

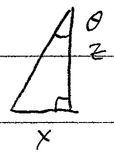
$$dE_{z}(x) = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{z^2+x^2} \cdot \cos\theta \cdot dx \cdot \lambda \,, \quad \cos\theta = \frac{z}{\sqrt{z^2+x^2}}$$

$$\therefore E_z = \int_{-L}^{L} \frac{1}{4\pi\epsilon_0} \frac{\lambda}{z^2+x^2} \cdot \frac{z}{\sqrt{z^2+x^2}} \, dx = \frac{z\lambda}{4\pi\epsilon_0} \int_{-L}^{L} \frac{1}{(z^2+x^2)^{3/2}} dx$$

$$= \frac{z\lambda}{4\pi\epsilon_0} \cdot 2 \underbrace{\int_0^L \frac{dx}{(z^2+x^2)^{3/2}}}_{\equiv I}$$

$$I \equiv \int_0^L \frac{dx}{(z^2+x^2)^{3/2}} = \int_0^{\tan^{-1}\frac{z}{L}} \frac{z\sec^2\theta \, d\theta}{z^3 \sec^3\theta} = \frac{1}{z^2} \int_0^{\tan^{-1}\frac{z}{L}} \cos\theta \, d\theta$$

$$= \frac{1}{z^2} \cdot \frac{L}{\sqrt{z^2+L^2}}$$



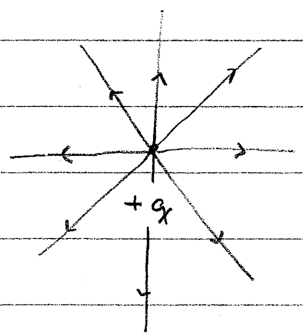
$$\begin{cases} z\tan\theta = x \\ z\sec^2\theta \, d\theta = dx \\ \tan^2\theta + 1 = \sec^2\theta, \quad \sec^2\theta = \frac{1}{\cos^2\theta} \end{cases}$$

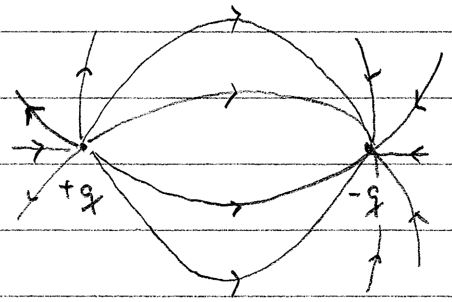
$$\therefore E_z = \frac{z\lambda}{4\pi\epsilon_0} 2 \frac{L}{z^2\sqrt{z^2+L^2}} = \frac{1}{4\pi\epsilon_0} \frac{2\lambda L}{z\sqrt{z^2+L^2}}$$

correct unit? ($\propto q/r^2$?)

2.2 Divergence and curl of eletrostatic fields

- Field Lines, flux, and Gauss's Law

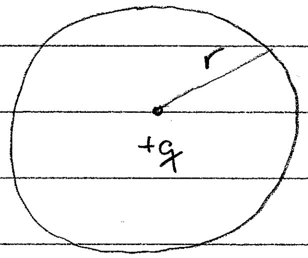




· The flux of $\vec{E}$ through a surface S

$$\Phi_E \equiv \int_S \vec{E}\cdot d\vec{a}$$

and let's compute it for a point charge:

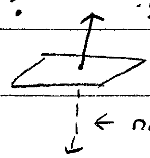


In spherical coord., infinitesimal area component

$$da = r^2 \sin\theta \, d\theta \, d\phi$$

Q: direction of da? · so $\hat{r}$!

→ We choose



← not this.

(why?) right hand rule...

$$\therefore \; d\vec{a} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\Phi_E = \oint_S \vec{E}\cdot d\vec{a}$$

$$= \iint \left(\frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \right) \cdot r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$= \frac{q}{4\pi\epsilon_0} \underbrace{\int_0^{\pi}\int_0^{2\pi} \sin\theta \, d\theta \, d\phi}_{\text{area of sphere w/ } r=1 \to 4\pi} = \frac{q}{4\pi\epsilon_0}\cdot 4\pi = \frac{q}{\epsilon_0}$$

So basically flux = charge / ϵ_0 !

↓ big step

This will be true for any closed surface!

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

: Gauss's law

To see its differential form, let's use divergence theorem

$$\oint_S \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E} \, d\tau \, , \quad Q_{enc} = \int_V \rho \, d\tau$$

So Gauss's law becomes

$$\int_V \vec{\nabla}\cdot\vec{E} \, d\tau = \frac{1}{\epsilon_0}\int_V \rho \, d\tau \quad \text{for any volume} \to \quad \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

• Direct calculation of $\vec{E}$

starting w/ $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\epsilon_0}\displaystyle\int_V \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\rho(\vec{r}')\,d\tau'$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0}\int_V \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\rho(\vec{r}')\,d\tau'$$

Using $\vec{\nabla}\cdot\left(\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\delta^3(\vec{\mathscr{r}})$

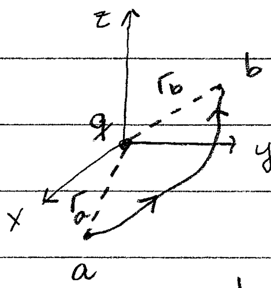
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0}\int_V 4\pi\delta^3(\vec{r}-\vec{r}')\rho(\vec{r}')\,d\tau' = \frac{1}{\epsilon_0}\rho(\vec{r})$$

• Gaussian surface : use symmetry!

• The curl of $\vec{E}$

For a point charge at the origin:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$



Suppose we calculate $\displaystyle\int_a^b \vec{E}\cdot d\vec{l}$

In spherical coord., $d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$$\therefore\ \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$$

So, $\displaystyle\int_a^b \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\epsilon_0}\int_a^b \frac{1}{r^2}\,dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$

The line integral of a closed path:

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = \left(\frac{1}{r_a} - \frac{1}{r_a}\right) = 0\ !$$

$$\oint_\Gamma \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{a}$$

$= 0$

$\therefore\ \vec{\nabla}\times\vec{E} = 0$

(true only for electrostatics)

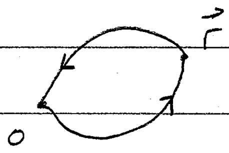
Using Stoke's theorem,

$$\int_S (\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \oint_\Gamma \vec{v}\cdot d\vec{l}$$

surface perimeter ([illegible])

- Electric Potential

$\vec{\nabla} \times \vec{E} = 0$ means $\oint \vec{E} \cdot d\vec{l} = 0$ for <u>any</u> closed loop



So there must be a unique function

$$V(\vec{r}) \equiv -\int_{0}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

this is called electric potential

$$V(\vec{b}) - V(\vec{a}) = -\int_{0}^{\vec{b}} \vec{E} \cdot d\vec{l} + \int_{0}^{\vec{a}} \vec{E} \cdot d\vec{l}$$

$$= -\int_{0}^{\vec{b}} \vec{E} \cdot d\vec{l} - \int_{\vec{a}}^{0} \vec{E} \cdot d\vec{l} = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

Using fundamental theorem for gradients

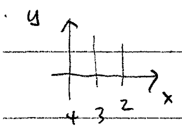
$$V(\vec{b}) - V(\vec{a}) = \int_{a}^{b} (\vec{\nabla} V) \cdot d\vec{l}$$

$$\int_{a}^{b} (\vec{\nabla} V) \cdot d\vec{l} = -\int_{a}^{b} \vec{E} \cdot d\vec{l} \quad \text{for any points } a, b$$

$$\therefore \vec{E} = -\vec{\nabla} V$$

- potential $\neq$ potential energy

∇V 의 기하학적의미



$V = 4 - x$

$\vec{\nabla} V = ?$

$V = 4 - 2x$

$\vec{\nabla} V = ?$

- Poisson's eq and Laplace's eq.

Let's get another form for electrostatic field

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}, \quad \vec{E} = -\vec{\nabla} V \qquad \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

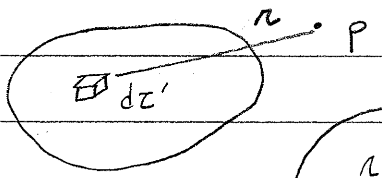
$$\rightarrow \vec{\nabla} \cdot (-\vec{\nabla} V) = -\nabla^2 V = +\frac{\rho}{\epsilon_0} \quad : \text{Poisson's eq.}$$

If there is no charge in space,

$$\nabla^2 V = 0 \quad : \text{Laplace's eq.}$$

$$\vec{\nabla} \times \vec{E} = \underbrace{\vec{\nabla} \times (-\vec{\nabla} V)}_{\text{always zero!}} = 0$$

The potential of a localized charge distribution



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{r}\,dq = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{r}\,d\tau'$$

r : distance from $d\tau'$ to point P.

So, this expression gives $V(\vec{r})$ for given $\rho(\vec{r})$

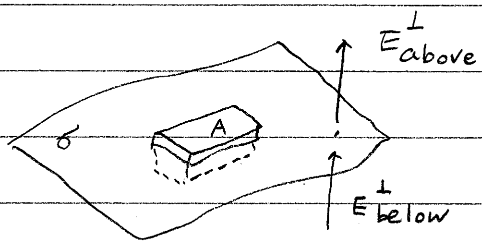
$\therefore$ this is the sol. to Poisson's equation

(well, if you can integrate above !)

- Boundary Conditions

We usually look for $\vec{E}$ and/or V for given $\rho(\vec{r})$.

Now, Suppose a thin charged layer exists and looks at $\vec{E}$:



Gauss's law says

$$\oint \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} = \frac{1}{\epsilon_0}\sigma A$$

and side does not contribute for small thickness, so

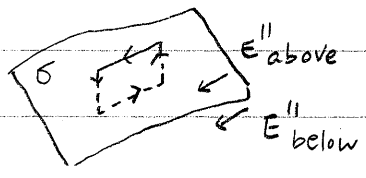
$$A\left(E^{\perp}_{above} - E^{\perp}_{below}\right) = \frac{1}{\epsilon_0}\sigma A$$

or

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$

: the normal component of $\vec{E}$ is discontinuous by σ/ϵ_0 at any boundary.

$E^{\parallel}$? From $\oint_{\Gamma} \vec{E}\cdot d\vec{l} = 0$, we get



$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0 \;!$$

$\therefore$ tangential comp. always continuous

Combining two: $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$

$\hat{n}$: unit vector, perpenticular to surface, pointing from below to above.

potential? Since $V_{above} - V_{below} = -\int_a^b \vec{E}\cdot d\vec{l}$

$V_{above} = V_{below}$ all the time.

But since $\vec{E} = -\vec{\nabla}V$,

$$\vec{\nabla}V_{above} - \vec{\nabla}V_{below} = -\sigma/\epsilon_0\,\hat{n}$$

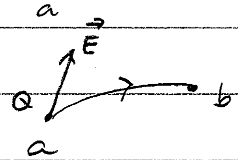
or

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\sigma/\epsilon_0 \quad \text{where } \frac{\partial V}{\partial n} = \vec{\nabla}V\cdot\hat{n}$$

called normal derivative

2.4 Work and Energy in Electrostatics

$W = \int_a^b \vec{F}\cdot d\vec{l}$ Work done by me to move Q from a to b ?



. electrostatic force $\vec{F} = Q\vec{E}$

. the force I have to apply is $\vec{F} = -Q\vec{E}$

$$\therefore\ W = -Q\int_a^b \vec{E}\cdot d\vec{l} = Q\left[V(b) - V(a)\right]$$

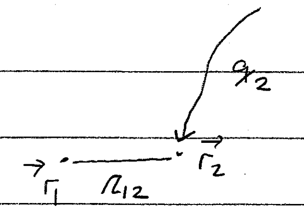
: path independent → conservative force!

Work I have to do bring Q from far away to $\vec{r}$

$$W = Q\left[V(\vec{r}) - V(\infty)\right]$$
$$= QV(\vec{r}) \quad (\text{setting } V(\infty) = 0)$$

The energy of a point charge distribution:

1) Bring q_1 from infinity : no work



Bring in q_2:

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{\mathscr{r}_{12}} \right)$$

Bring in q_3 at $\vec{r}_3$ on top of q_1 and q_2

$$W_3 = \frac{1}{4\pi\epsilon_0} q_3 \left(\frac{q_1}{\mathscr{r}_{13}} + \frac{q_2}{\mathscr{r}_{23}} \right)$$

And for q_4, $W_4 = \frac{1}{4\pi\epsilon_0} q_4 \left(\frac{q_1}{\mathscr{r}_{14}} + \frac{q_2}{\mathscr{r}_{24}} + \frac{q_3}{\mathscr{r}_{34}} \right)$

and so on...

$\therefore$ The total work to assemble four charges

$$W = W_2 + W_3 + W_4 = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{\mathscr{r}_{12}} + \frac{q_1 q_3}{\mathscr{r}_{13}} + \frac{q_2 q_3}{\mathscr{r}_{23}} + \frac{q_1 q_4}{\mathscr{r}_{14}} + \frac{q_2 q_4}{\mathscr{r}_{24}} + \frac{q_3 q_4}{\mathscr{r}_{34}} \right)$$

So the general rule for n charges:

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}}$$

$$= \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{\mathscr{r}_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \underbrace{\left(\sum_{j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{\mathscr{r}_{ij}} \right)}_{\text{potential at point } \vec{r}_i \text{ due to all other charges}}$$

$$= \frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)$$

- The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2} \int_V \rho V(\vec{r})\, d\tau$$ over the volume integral

Now, $\rho = \epsilon_0 \vec{\nabla} \cdot \vec{E}$ so $W = \frac{\epsilon_0}{2} \int_V (\vec{\nabla} \cdot \vec{E}) V \, d\tau$

From $\int \vec{\nabla} \cdot (f \vec{A}) d\tau = \int f (\vec{\nabla} \cdot \vec{A}) d\tau + \int \vec{A} \cdot (\vec{\nabla} f) d\tau = \oint_S f \vec{A} \cdot d\vec{a}$

or $\int f (\vec{\nabla} \cdot \vec{A}) d\tau = - \int \vec{A} \cdot (\vec{\nabla} f) d\tau + \oint_S f \vec{A} \cdot d\vec{a}$

$$W = \frac{\epsilon_0}{2}\left[-\int_V \vec{E}\cdot \overset{-\vec{E}}{\overset{\parallel}{(\vec\nabla V)}}\, d\tau + \oint_S V\,\vec{E}\cdot d\vec{a} \right]$$

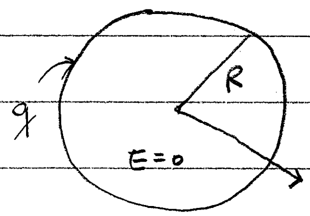
$$= \frac{\epsilon_0}{2}\left[\int_V E^2 d\tau + \underbrace{\oint_S V\vec{E}\cdot d\vec{a}} \right]$$

If we integrate over all space,

$$\oint \underbrace{V\vec{E}}_{\propto \frac{1}{r^3}}\cdot d\vec{a} = 0\,!$$

$$\therefore\ W = \frac{\epsilon_0}{2}\int E^2 d\tau \quad \text{(all space)}$$

Ex 2.9) Find the energy of a uniformly charged spherical shell of total charge q and radius R



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

$$W = \frac{\epsilon_0}{2}\int \frac{q^2}{(4\pi\epsilon_0)^2}\frac{1}{r^4}\cdot r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}\cdot(4\pi)\int_R^\infty \frac{1}{r^2}dr$$

$$= \frac{\epsilon_0}{2}\cdot\frac{q^2}{4\pi\,\epsilon_0^2}\cdot\frac{1}{R} = \frac{1}{8\pi\epsilon_0}\frac{q^2}{R}$$

2.5 Conductors

- electrons are free to move.

Basic properties of ideal conductor

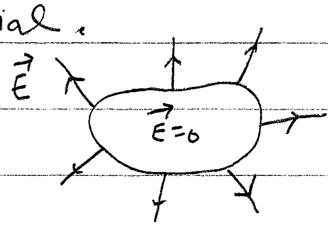
i) $\vec{E} = 0$ inside a conductor

ii) $\rho = 0$ inside a conductor $\because\ \vec\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$

iii) Any net charge resides on the surface

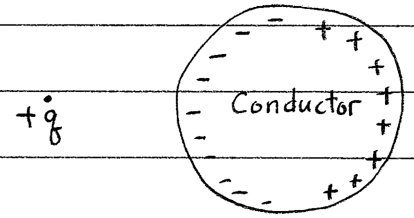
iv) A conductor is an equipotential.

v) $\vec{E} \perp$ surface

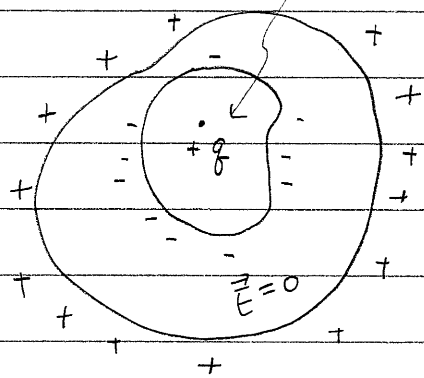


- Induced Charges

i)

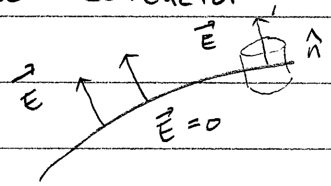


inside hollow $\vec{E} \neq 0$ due to Gauss's law.



- Surface charge and the Force on a Conductor

For a conductor, the field immediately outside:



(positively charged)

$$E \cdot dA = \frac{\sigma \, dA}{\epsilon_0}$$

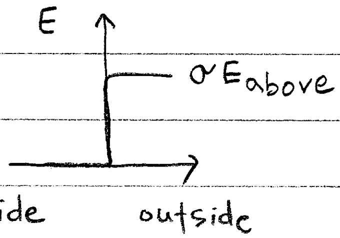
$$\downarrow \quad \vec{E} = \frac{\sigma}{\epsilon_0} \hat{n}$$

$\hat{n}$: unit vector outside

In terms of potential $\vec{\nabla} V = -\vec{E}$ gives us

$$E = \frac{\sigma}{\epsilon_0} = -\frac{\partial V}{\partial n} \rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

- Since $\vec{E}$ is present, a surface charge will experience a force / unit area ($\equiv \vec{f} = \sigma \vec{E}$). But



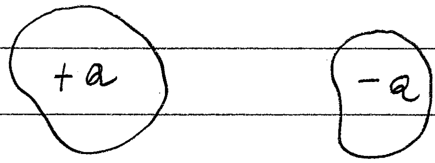
$$\therefore \vec{f} = \sigma \vec{E} = \frac{1}{2} \sigma \left(\vec{E}_{above} + \vec{E}_{below} \right)$$

$$\vec{f} = \frac{\sigma}{2} \cdot \frac{\sigma}{\epsilon_0} \hat{n} = \frac{1}{2\epsilon_0} \sigma^2 \hat{n}$$

$\therefore$ electrostatic pressure on the surface: $P = \frac{\epsilon_0}{2} E^2$

Capacitors

What is capacitors? – consider two conductors, $+Q$, $-Q$ at each



Potential difference between them?

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$$

What happens on V if $Q \to 2Q$, $-Q \to -2Q$?

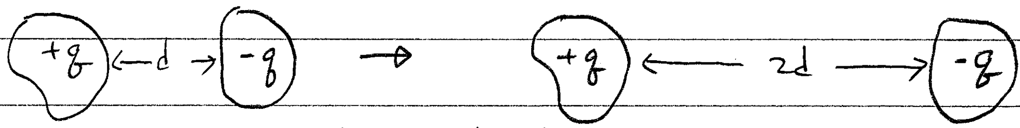
In general, $\vec{E} = \dfrac{1}{4\pi\epsilon_0} \dfrac{Q}{r^2} \hat{r}$ (point charge)

$\therefore\ E \propto Q \longrightarrow V \propto Q \qquad \therefore\ \dfrac{Q}{V}$ must be a constant!

We define the capacitance as

$$C \equiv \frac{Q}{V}$$

Let's imagine of separating them further:



$E \to$ reduced by $\dfrac{1}{(2d)^2} = \dfrac{1}{4} \cdot \dfrac{1}{d^2}$

$\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l}$: distance increased by 2d

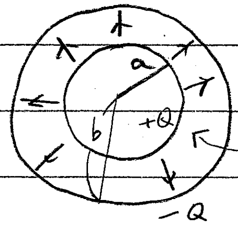
$\hookrightarrow$ C is reduced by 2 times! (?)

$\therefore$ C depends on geometry of the System we consider.

SI unit: Farad (F) = Coulomb / Volt

1 F: too large (μF, pF are more common)

Ex 2.12 Find the capacitance of two concentric spherical metal shells with radii a and b.



Let's put $+Q$ on inner shell
$-Q$ " outer "

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

∴ potential difference

$$V = -\int_b^a \vec{E} \cdot d\vec{l} = -\frac{Q}{4\pi\epsilon_0} \int_b^a \frac{1}{r^2} dr$$

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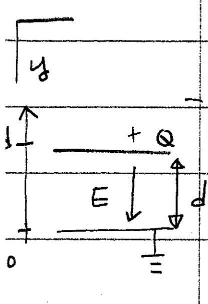
$$= \frac{Q}{4\pi\epsilon_0} \frac{1}{r}\Big|_b^a = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\epsilon_0} \frac{b-a}{ab}$$

$$C = Q/V = 4\pi\epsilon_0 \frac{ab}{(b-a)}$$



$V = ?$

$$V = -\int_d^0 \vec{E} \cdot d\vec{l}$$

$$= -\int_d^0 E\,dy$$

$= Ed$
(>0)

· Work to charge up a capacitor.

$$W = QV \rightarrow dW = V(q)\,dq = \left(\frac{q}{C}\right) dq$$

$$W = \int_0^Q \left(\frac{q}{C}\right) dq = \frac{1}{2}\frac{Q^2}{C} \quad \text{or} \quad W = \frac{1}{2} C V^2$$