Forced Oscillation and Resonance

Chapter 02

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Dampled Oscillators

- Damped Oscillators

We have a SHO with a damping term, represented as

$$-m\Gamma v$$

where Γ and v are a constant and the velocity of an object, respectively.

Question: what should be dimension of Γ in order for the above term to be intepreted as force?
Ans.

$$[\Gamma] = time^{-1}$$

The equation of motion of such system is

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = 0.$$

Note that $\omega_0 = \sqrt{K/m}$.

Damped Oscillators

Now we allow for the possibility of complex-valued solutions. Namely

$$\frac{d^2}{dt^2}z(t) + \Gamma\frac{d}{dt}z(t) + \omega_0^2 z(t) = 0.$$

Note that is is possible since coefficients such as Γ and ω_0^2 are real-valued. One possible way to systematically solve it is assume the solution to be

$$z(t) = e^{\alpha t}$$

where α is a complex-valued constant. Having it into the differential equation gives

$$(\alpha^2 + \Gamma\alpha + \omega_0^2)e^{\alpha t} = 0$$

so

$$\alpha = -\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$$

For $ax^2 + bx + c = 0$, $x = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - ac}$.

Damped Oscillations

must be satisfied. We will see that there are three regions for Γ compared to ω_0.

- Overdamped Oscillators : $\Gamma/2 > \omega_0$

If $\Gamma/2 > \omega_0$, then $\Gamma^2/4 - \omega_0^2 > 0$. Therefore, both solutions for α are real and negative in this case. So the solution is a sum of **decreasing exponentials** in general. Any initial displacement of the system dies away with no oscillation: an **overdamped oscillator**.
The general solution in this case is

$$x(t) = z(t) = A_+ e^{-\Gamma_+ t} + A_- e^{-\Gamma_- t}$$

where

$$\Gamma_\pm \equiv \frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$$

and $A_\pm$ are amplitudes to be determined by initial conditions.

A Damped Oscillator: Example

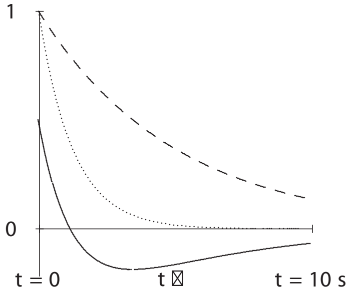


Figure 2.1: Solutions to the equation of motion for an overdamped oscillator.

Dotted line is $e^{-\Gamma_+ t}$ for $\Gamma = 1\ \mathrm{s}^{-1}$ and $\omega_0 = 4\ \mathrm{s}^{-1}$. The dashed line is $e^{-\Gamma_- t}$. The solid line is a linear combination, $e^{\Gamma_+ t} - \frac{1}{2} e^{-\Gamma_- t}$.

Damped Oscillations

- Underdamped Oscillators : $\Gamma/2 < \omega_0$

The solutions for α are a complex conjugate pair, with negative real part. With

$$\omega^2 \equiv \omega_0^2 - \Gamma^2/4, \longrightarrow \alpha = -\frac{\Gamma}{2} \pm i\omega$$

This is an underdamped oscillator. So $z(t)$ can be in a form of

$$z(t) = e^{-\Gamma t/2}\left(\alpha e^{-i\omega t} + \beta e^{i\omega t}\right), \quad \alpha, \beta \in \mathbb{C}$$

If we expand exponential term and rearrange, we get

$$z(t) = e^{-\Gamma t/2}\left(c_1 \cos \omega t + c_2 \sin \omega t\right), \quad c_1, c_2 \in \mathbb{C}$$

$$x(t) = Re(z(t)) = e^{-\Gamma t/2}\left(c \cos(\omega t) + d \sin(\omega t)\right), \quad c, d \in \mathbb{R}.$$

Damped Oscillations

- Critically Damped Oscillators : $\Gamma/2 = \omega_0$

If $\Gamma/2 = \omega_0$, $\alpha = -\Gamma/2$, $x(t) \propto e^{-\Gamma t/2}$. But there must be two solutions since we deal with 2nd order differential equations.

How do we get the 2nd solution from the first solution $e^{-\Gamma t/2}$? One (may look strange) way is

$$\lim_{\omega \to 0} \frac{1}{\omega} e^{-\Gamma t/2} \sin \omega t = \lim_{\omega \to 0} e^{-\Gamma t/2} \frac{1}{\omega} \left(\omega t - \frac{1}{3!} (\omega t)^3 + ... \right) = t \ e^{-\Gamma t/2}.$$

Therefore the 2nd solution is $t \ e^{\Gamma t/2}$, and this is for the critically damped case. Now, you may ask yourself...

Damped Oscillations

- Critically Damped Oscillators : $\Gamma/2 = \omega_0$ (cont.)

$x(t) = te^{-\Gamma t/2}$ is really the 2nd solution?

$$\frac{d}{dt}x(t) = e^{-\Gamma t/2} - \frac{\Gamma t}{2}e^{-\Gamma t/2}$$
$$\frac{d^2}{dt^2}x(t) = -\frac{\Gamma}{2}e^{-\Gamma t/2} - \frac{\Gamma}{2}e^{-\Gamma t/2} + \left(\frac{\Gamma}{2}\right)^2 te^{-\Gamma t/2}$$

Now since

$$\begin{aligned}
&\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) \\
&= -\Gamma e^{-\Gamma t/2} + \left(\frac{\Gamma}{2}\right)^2 te^{-\Gamma t/2} + \Gamma e^{-\Gamma t/2} - \frac{\Gamma^2 t}{2}e^{-\Gamma t/2} + \frac{\Gamma^2}{4}te^{-\Gamma t/2} \\
&= 0,
\end{aligned}$$

$x(t) = te^{-\Gamma t/2}$ is indeed the 2nd solution. Therefore, in general solution in the critically damped case is

$$x(t) = c\ e^{-\Gamma t/2} + d\ te^{-\Gamma t/2}.$$

Forced Oscillations

The damped oscillator with a harmonic driving force has the following equation of motion:

$$\frac{d^2}{dt^2}x(t) + \Gamma\frac{d}{dt}x(t) + \omega_0^2 x(t) = \frac{F(t)}{m}$$

where the force itself can be written as

$$F(t) = F_0 \cos \omega_d t \qquad \frac{\omega_d}{2\pi} : \text{ driving frequency}$$

We now relate above to an equation of motion with a complex driving force

$$\frac{d^2}{dt^2}z(t) + \Gamma\frac{d}{dt}z(t) + \omega_0^2 z(t) = \frac{\mathcal{F}(t)}{m}$$
$$\mathcal{F}(t) = F_0 e^{-i\omega_d t}, \qquad Re\mathcal{F}(t) = F(t).$$

Forced Oscillators

A steady state solution (*a form of the solution when time is spent enough so that the response of the system is mainly due to the driving force*) of the form can be written as

$$z(t) = \mathcal{A} e^{-i\omega_d t}.$$

This is the form that survives for a long time in the present of damping (nothing to do with the initial conditions). Now, having $z(t) = \mathcal{A} e^{-i\omega_d t}$ to the differential equation gives us

$$(-\omega_d^2 - i\Gamma\omega_d + \omega_0^2)\mathcal{A} = \frac{F_0}{m}$$

$$\mathcal{A} = \frac{F_0/m}{\omega_0^2 - \omega_d^2 - i\Gamma\omega_d}.$$

Therefore, $\mathcal{A}$ is proportional to the amplitude of the driving force (consistent with our intuition).

Forced Oscillators

Let's make the denominator real:

$$\begin{aligned}\mathcal{A} &= \frac{(\omega_0^2 - \omega_d^2 + i\Gamma\omega_d)F_0/m}{(\omega_0^2 - \omega_d^2)^2 + \Gamma^2\omega_d^2} \\ &= \underbrace{\frac{(\omega_0^2 - \omega_d^2)F_0/m}{(\omega_0^2 - \omega_d^2)^2 + \Gamma^2\omega_d^2}}_{\equiv A} + i\underbrace{\frac{\Gamma\omega_d F_0/m}{(\omega_0^2 - \omega_d^2)^2 + \Gamma^2\omega_d^2}}_{\equiv B} \\ &= A + iB.\end{aligned}$$

Then the solution to the equation of motion for the real driving force is

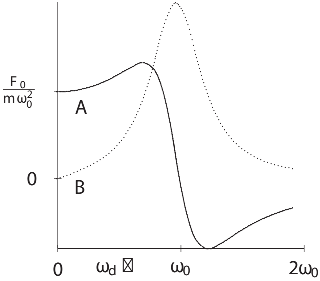
$$\begin{aligned}x(t) = Re(z(t)) = Re(\mathcal{A}e^{-i\omega_d t}) &= Re\Big[(A + iB)(\cos\omega_d t - \sin\omega_d t)\Big] \\ &= A\cos\omega_d t + B\sin\omega_d t.\end{aligned}$$

The term proportional to A: in phase with the driving force, and the term proportional to B: 90° out of phase with the driving force.

Forced Oscillators

The advantage of going to the complex driving force is that it allows us to get both solutions at once.

The coefficients, A and B, are shown below for $\Gamma = \omega_0/2$.



The real part of $\mathcal{A}$, A is call the **elastic amplitude** (solid line) and the imaginary part, B is called the **absorptive amplitude** (dotted line) in the figure above.

Resonance

At $\omega_d = \omega_0$, the response of the system to the driving force increases ($B \gg 1, A = 0$) and the system is said to be **on resonance**. Very often, we will ignore damping in forced oscillations: near a resonance, this is not a good idea because $A, B \to \infty$ as $\Gamma \to 0$ for $\omega_d = \omega_0$.

Work

Work done by the external force to the system?

$$\begin{aligned} P(t) &= F(t) \cdot \frac{d}{dt} x(t) \\ &= (F_0 \cos \omega_d t) \cdot \frac{d}{dt}(A \cos \omega_d t + B \sin \omega_d t) \\ &= -F_0 A \omega_d \cos \omega_d t \sin \omega_d t + F_0 \omega_d B \cos^2 \omega_d t \end{aligned}$$

The 1st term is proportional to $\sin \omega_d t$ and averaging over any complete half-period of oscillation which is $\left(\frac{2\pi}{\omega_d}\right)\frac{1}{2}$

$$\begin{aligned} \int_{t_0}^{t_0+\pi/\omega_d} dt \;\; \sin 2\omega_d t &= -\frac{1}{2} \cos 2\omega_d t \Bigg|_{t_0}^{t_0+\pi/\omega_d} \\ &= -\frac{1}{2}\Big(cos(2\omega_d t_0 + 2\pi) - \cos 2\omega_d t_0 \Big) \\ &= 0 \end{aligned}$$

and this is why A is called the elastic amplitude.

Work

If A dominates, then the energy fed into the system at one time is returned. The 2nd term $F_0 B \cos^2 \omega_d t$ is always positive (note $B > 0$) and

$$P_{average} = \frac{1}{2} F_0 \omega_d B$$

is satisfied. This is why B is called absorptive amplitude. $P_{average}$ reaches maximum on resonances, at $\omega_d = \omega_0$ (Is it obvious?). At any rate,

$$P_{average} = \frac{1}{2} F_d \omega_d \frac{\Gamma \omega_d F_0 / m}{(\omega_0^2 - \omega_d^2)^2 + \Gamma^2 \omega_d^2}$$

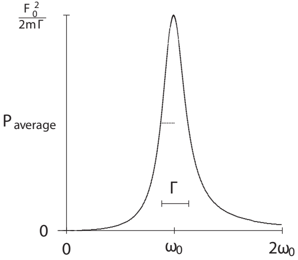
$$at \quad \omega_D = \omega_0,$$

$$P_{average} = \frac{1}{2} F_0 \omega_d \frac{\Gamma \omega_d F_0 / m}{\Gamma^2 \omega_d^2} = \frac{F_0^2}{2m\Gamma}.$$

Resonance Width and Lifetime

The maximum average power is inversely proportional to Γ:

$$P_{average}(\omega_d = \omega_0) = \frac{F_0^2}{2m\Gamma}$$



One can check that values of ω_d for which the average power loss is half its maximum value are:

$$w_{1/2} = \sqrt{\omega_0^2 + \frac{\Gamma^2}{4}} \pm \frac{\Gamma}{2}$$

Resonance Width and Lifetime

Proof:

$$\underbrace{\frac{F_0^2}{2m\Gamma} \times \frac{1}{2}}_{power \times \frac{1}{2}} = \underbrace{\frac{1}{2} F_0 \omega_{1/2} \cdot \frac{\Gamma \omega_{1/2} F_0/m}{(\omega_0^2 - \omega_{1/2}^2)^2 + \Gamma^2 \omega_{1/2}^2}}_{power\ at\ \omega_d = \omega_{1/2}}$$

$$\rightarrow (\omega_0^2 - \omega_{1/2}^2)^2 + \Gamma^2 \omega_{1/2}^2 = 2\omega_{1/2}^2 \Gamma^2$$

$$\rightarrow (\omega_0^2 - \omega_{1/2}^2)^2 - \Gamma^2 \omega_{1/2}^2 = 0$$

$$Using\ A^2 - B^2 = 0 \rightarrow (A - B)(A + B) = 0,$$

$$\rightarrow (\omega_0^2 - \omega_d^2 - \Gamma \omega_{1/2})(\omega_0^2 - \omega_d^2 + \Gamma \omega_{1/2}) = 0$$

$$so,\ either\ \omega_0^2 - \omega_d^2 - \Gamma \omega_{1/2} = 0$$

$$or \omega_0^2 + \omega_d^2 - \Gamma \omega_{1/2} = 0\ \ must\ be\ satisfied.$$

$$From\ the\ first,\ \omega_{1/2} = -\frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \omega_0^2} \quad (\because \omega_{1/2} > 0)$$

$$from\ the\ second,\ \omega_{1/2} = \frac{\Gamma}{2} + \sqrt{\frac{\Gamma^2}{4} + \omega_0^2} \quad (\because \omega_{1/2} > 0)$$

$$\therefore \omega_{1/2} = \sqrt{\frac{\Gamma^2}{4} + \omega_0^2} \pm \frac{\Gamma}{2}.$$

Phase Lag

$x(t) = A\cos\omega_d t + B\sin\omega_d t$ can be written as

$$x(t) = R\cos(\omega_d t - \theta)$$

for $R = \sqrt{A^2 + B^2}$, $\theta = arg(A + iB)$. Note that θ is called phase lag, measuring the phase lag between the external force and the system's response (the actual time lag is θ/ω_d: why? think about the unit!).