Wave Physics

Chapter 01

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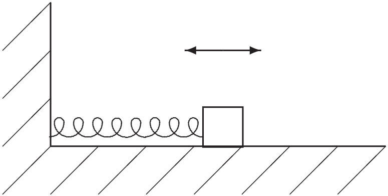
Auguest 1, 2011

Simple Harmonic Motion

- The Harmonic Oscillator

Oscillations are the basic building blocks of waves. We begin by discussing the harmonic oscillator.



This system has only one relevant degree of freedom. In general, the number of **degrees of freedom** of a system is the number of coordinates that must be specified in order to determine the configuration completely.

Simple Harmonic Motion

The force on the block takes the form of

$$F = -Kx$$

where K is called "spring constant". The Newton's law implies

$$m\frac{d^2}{dt^2}x(t) = -Kx(t).$$

Mathematically, this is called a second-order, ordinary differential equation. The most general solution to this differential equation of motion is a sum of a constant times $\cos\omega t$ plus a constant times $\sin\omega t$.

$$x(t) = a\cos\omega t + b\sin\omega t,$$

$$\omega \equiv \sqrt{\frac{K}{m}}.$$

Simple Harmonic Motion

The ω is a constant with units of T^{-1} called the "angular frequency". The values of position and velocity at $t = 0$ are called **initial conditions**. For example, we can write the **most general solution** in terms of $x(0)$ and $x'(0)$, the displacement and velocity of the block at time $t = 0$. Setting $t = 0$ givea $a = x(0)$. Differentiating and then setting $t = 0$ gives $b = \omega x'(0)$. Thus

$$x(t) = x(0)\cos\omega t + \frac{1}{\omega}x'(0)\sin\omega t,$$

The motion is periodic, and after a time

$$\tau = \frac{2\pi}{\omega}$$

the system returns exactly to where it was at $t = 0$. The time τ (Greek letter tau) is called the "period" of the oscillation. The frequency ν, is the inverse of the period τ

$$\nu = \frac{1}{\tau}$$

Small Oscillations and Linearity

A system with one degree of freedom is **linear** if its equation of motion is a linear function of the coordinate, x, that specifies the system's configuration. The equation of motion involves a second derivative, but no higher derivative, so a linear equation of motion has the general form:

$$\alpha \frac{d^2}{dt^2} x(t) + \beta \frac{d}{dt} x(t) + \gamma x(t) = f(t)$$

. If all the terms involve exactly one power of x, the equation of motion is "homogeneous." The inhomogeneous term $f(t)$, represents an external force. We will assume that α, β, and γ are real-valued constants.

Time Translation Invariance

When α, β, and γ do not depend on the time, t, and in the absence of an external force, times enters only through derivatives. Then

$$\alpha\frac{d^2}{dt^2}x(t)+\beta\frac{d}{dt}x(t)+\gamma x(t)=0.$$

The equation of motion for the undamped harmonic oscillator has this form with $\alpha = m$, $\beta = 0$, $\gamma = K$.
The solutions to the above equation have the property that

If $x(t)$ is a solution, $x(t+a)$ will be a solution also.

Mathematically, this is true because the operation with respect to time and replacing $t \to t+a$ can be done in either order because of the chain rule

$$\frac{d}{dt}x(t+a)=\left[\frac{d}{dt}(t+a)\right]\left[\frac{d}{dt'}x(t')\right]_{t'=t+a}=\left[\frac{d}{dt'}x(t')\right]_{t'=t+a}.$$

The physical reason is that we can change the initial setting on our clock and the physics will look the same.

Complex Numbers

We define

$$i = \sqrt{-1}$$

so i is a number (called complex number) that gives us -1 when we square it. In general, a complex number z, is a sum of a real number and an imaginary number: $z = a + ib$: the real part, $\mathrm{Re}(z) = a$, and the imaginary part, $\mathrm{Im}(z)=b$ are both real numbers.

The complex conjugate z^*, of the complex number $z = a + ib$ is

$$z^* = a - ib.$$

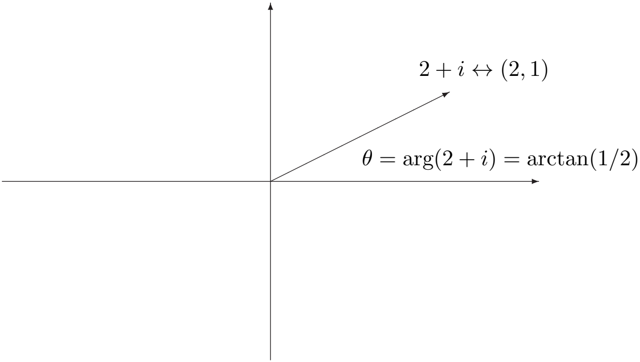
Note that $\mathrm{Re}(z) = (z + z^*)/2$ and $\mathrm{Im}(z) = (z - z^*)/2i$.

The complex plane: Because a complex number z is specified by two real numbers, it can be thought of as a two-dimensional vector, with components (a, b).

The absolute value, $|z|$, of z, is the length of the vector (a, b):

$$|z| = \sqrt{a^2 + b^2} = \sqrt{z^* z}$$

Complex Numbers



The argument or phase: $\arg(z)$, of a nonzero complex number z, is the angle, in radians, of the vector (a, b) counterclockwise from the x axis:

$$arg(z) = \begin{cases} \arctan(b/a) \text{ for } a \geq 0, \\ \arctan(b/a) + \pi \text{ for } a < 0. \end{cases}$$

Complex Numbers - Arithmetic

If $z = a + ib$ and $z' = a' + ib'$, then

$$z + z' = (a + a') + i(b + b'),$$

$$z - z' = (a - a') + i(b - b'),$$

$$zz' = (aa' - bb') + i(ab' + ba').$$

$$|zz'| = |z||z'|$$

$$|z/z'| = |z|/|z'| \text{ if } z' \neq 0.$$

Complex Exponentials

Consider a complex number $z = a + ib$ with absolute value 1. Because $|z| = 1$ implies $a^2 + b^2 = 1$, we can write

$$z = \cos\theta + i\sin\theta \quad \text{for} \quad |z| = 1.$$

Because

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b}{a}$$

the angle θ is the argument of z:

$$\arg(\cos\theta + i\sin\theta) = \theta.$$

We can also relates to the exponential:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where we can also write as

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Complex Exponentials

In more general,

$$z = x + iy = Re^{i\theta} \quad \text{where} \quad R = |z|, \quad \text{and} \quad \theta = \arg(\mathrm{z}).$$

So,

$$z_1 = R_1 e^{i\theta_1} \quad \text{and} \quad z_2 = R_2 e^{i\theta_2},$$

then

$$z_1 z_2 = R_1 R_2 e^{i(\theta_1 + \theta_2)}.$$

Exponential Solutions

We have seen that the solutions of homogeneous linear differential equations with constant coefficients, of the form

$$\mathcal{M}\frac{d^2}{dt^2}x(t) + \mathcal{K}x(t) = 0$$

have the properties of linearity and time translation invariance. The equation of simple harmonic motion is of this form. If it describes a physics problem, the coordinates are real, and the constants $\mathcal{M}$ and $\mathcal{K}$ are real because they should correspond quantities such as mass and spring constants.

However, let's consider

$$\mathcal{M}\frac{d^2}{dt^2}z(t) + \mathcal{K}z(t) = 0$$

where a real-valued function is replaced with a complex-valued function $z(t)$.

Exponential Solutions

Because the coefficients $\mathcal{M}$ and $\mathcal{K}$ are real, for every solution, $z(t)$, the complex conjugate, $z(t)^*$, is also a solution (Can you prove?). From these two solutions, we can construct two real solutions:

$$x_1(t) = \mathrm{Re}(z(t)) = (z(t) + z(t)^*)/2,$$

$$x_2(t) = \mathrm{Im}(z(t)) = (z(t) - z(t)^*)/2i.$$

Now a question arises: why do we bother to use complex variables to solve physical system that does not know complex numbers? That's because **it is much easier to solve with complex variables and at the end you get the physical solution of interest by taking the real part of your complex solution.**