1. $(A+A^{\dagger})^{\dagger} = A^{\dagger}+A$

$[\,i(A-A^{\dagger})\,]^{\dagger} = -i\,(A^{\dagger}-A) = i\,(A-A^{\dagger})$

2 $y'' = (x^2-E)\,y$

a) $y(x) = h(x)\, e^{-x^2/2}$

$y' = (h' - xh)\, e^{-x^2/2}$

$y'' = (h'' - 2xh' + (x^2-1)h)\, e^{-x^2/2}$

This gives

$$h'' - 2xh' + (E-1)h = 0$$

b) $h(x) = \sum_{j=0}^{\infty} a_j x^j$

$h' = \sum_{j=0}^{\infty} j a_j x^{j-1}$ $\qquad h'' = \sum_{j=0}^{\infty} (j+1)(j+2)\, a_{j+2}\, x^j$

$$\rightarrow \sum_{j=0}^{\infty} \left[(j+1)(j+2)\, a_{j+2} - 2j a_j + (E-1)\, a_j \right] x^j = 0$$

$$\rightarrow \quad (j+1)(j+2)\, a_{j+2} - 2j a_j + (E-1)\, a_j = 0$$

$$\therefore \quad a_{j+2} = \frac{(2j+1-E)}{(j+1)(j+2)}\, a_j$$

c) $E = 2n+1$

$$a_{j+2} = \frac{-2(n-j)}{(j+1)(j+2)}\, a_j$$

$n=0$: $h_0(x) = a_0$ $\qquad y_0(x) = a_0\, e^{-x^2/2}$

$n=1$ $\quad h_1(x) = a_1 x$ $\qquad y_1(x) = a_1 x\, e^{-x^2/2}$

$n=2$ $\quad h_2(x) = a_0(1-2x^2)$ $\quad y_2(x) = a_0\,(1-2x^2)\, e^{-x^2/2}$

(you just solved simple harmonic oscillator problem in Quantum Mechanics)

3 a) $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} - \frac{\partial^2 u}{\partial t^2} = 0$

$$\rightarrow \frac{\partial^2 u}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial\phi^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial^2 u}{\partial\phi^2} = -n^2 u$$

the above partial diff. eq becomes

$$\frac{\partial^2 f_n}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial f_n}{\partial\rho} - \frac{n^2}{\rho^2} f_n - \frac{\partial^2 f_n}{\partial t^2} = 0$$

b) $f_n(\rho, t) = \sum_{j=1}^{\infty} g_{jn}(t)\, h_n(\rho)$

$$\rightarrow g_{jn}(t)\left[\frac{d^2 h_n}{d\rho^2} + \frac{1}{\rho}\frac{dh_n}{d\rho} - \frac{n^2}{\rho^2} h_n\right] - \frac{d^2 g_{jn}}{dt^2}\cdot h_n = 0$$

c) If $\frac{d^2 h_n}{d\rho^2} + \frac{1}{\rho}\frac{dh_n}{d\rho} - \frac{n^2}{\rho^2} h_n = -k_{jn}^2 h_n$

① $k_{jn}^2 g_{jn}(t) + \frac{d^2 g_{jn}}{dt^2} = 0 \rightarrow g_{jn}(t) = e^{\pm i k_{jn} t}$

② we have $\frac{d^2 h_n}{d\rho^2} + \frac{1}{\rho}\frac{dh_n}{d\rho} + \left(k_{jn}^2 - \frac{n^2}{\rho^2}\right) h_n = 0$

This is Bessel's ODE and the boundary cond. requires us to pick J_n.

$$\therefore\ u(r, \theta, t) = \sum_{j,n}^{\infty} A_{jn} J_n(k_{jn}\rho)\left(\sin n\phi + B_n \cos n\phi\right) \times e^{\pm i \omega_{jn} t}$$

4. I showed it explicitly in the class (see the lecture note)

5. Let $a=1$ in the problem 4 and use the orthogonality then we get

$$\int_0^1 [f(x)]^2 x\, dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 \left[J_{n+1}(a_{mn})\right]^2$$