# Mathematical Physics II, Mid-term Exam **by E. Won**

1. $A$ is a non-Hermitian operator. Show that

$$A + A^\dagger \qquad \text{and} \qquad i(A - A^\dagger)$$

are Hermitian operators.

2. Let us solve the below ODE step by step.

$$y''(x) = (x^2 - E)y(x)$$

**a)** Assume

$$y(x) = h(x)e^{-x^2/2}$$

and obtain a new ODE for $h(x)$.

**b)** Now, assume

$$h(x) = \sum_{j=0}^{\infty} a_j x^j$$

and insert it into the ODE for $h(x)$ obtained in a). Derive the relation between $a_{j+2}$ and $a_j$.

**c)** The series solution of $h(x)$ we assume in b) diverges. Therefore, the power series must terminate. So for a certain integer $n$, $E = 2n+1$ has to be satisfied. Using it, find the solutions of the original ODE for $y(x)$ when $n$=0,1, and 2, respectively.

3. Consider the following differential equation

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

where $u(\rho, \phi, t)$ describes the drum-head vibration.

**a)** Assume

$$u = \sum_{n=0}^{\infty} f_n(\rho, t)(\sin n\phi + a_n \cos n\phi)$$

and obtain the new differential equation for $f_n(\rho, t)$.

**b)** Assume

$$f_n(\rho, t) = \sum_{j=1}^{\infty} g_{jn}(t) h_n(\rho)$$

and obtain the new differential equation.

**c)** Use the *separation of variables* technique to obtain two ODEs and identify $g_{jn}$ and $h_n(\rho)$. Note: the boundary condition is that $u(\rho = 0)$ is finite and $u(\rho{=}R){=}0$ so choose your solutions carefully.

4. Show the below relation.

$$\int_0^a \left[J_\nu\left(\alpha_{\nu m}\frac{\rho}{a}\right)\right]^2 \rho\, d\rho = \frac{a^2}{2}[J_{\nu+1}(\alpha_{\nu m})]^2$$

where $J_\nu(\alpha_{\nu m})$ is the Bessel function and $\alpha_{\nu m}$ is the $m$th zero of $J_\nu$.

5. A function $f(x)$ is expressed as a Bessel series:

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(\alpha_{mn} x),$$

with $\alpha_{mn}$ the $n$th root of $J_m$. Prove the following Parseval relation,

$$\int_0^1 [f(x)]^2 x\, dx = \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 [J_{m+1}(\alpha_{mn})]^2.$$

You may use the result of the problem 4.