1. $g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\ e^{ikx} f(x)$

$\Rightarrow \frac{-D}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \frac{d^2\phi(x)}{dx^2} e^{ikx} + \frac{K^2 D}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \phi(x) e^{ikx} = \frac{Q}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, \delta(x)$

$\Rightarrow Dk^2 g(k) + DK^2 g(k) = \frac{Q}{\sqrt{2\pi}}$

$g(k) = \frac{Q}{\sqrt{2\pi}\, D(k^2+K^2)}$

so, $\phi(x) = \frac{Q}{D \cdot 2\pi} \int_{-\infty}^{\infty} dk \frac{e^{-ikx}}{k^2+K^2}$

2. $f(x) = \begin{cases} 0 & -1<x<0 \\ x & 0<x<0 \end{cases}$

$f(x) = \sum_{l=0}^{\infty} c_l P_l(x)$

$\int_{-1}^{1} f(x) P_m(x) dx = \sum_{l=0}^{\infty} c_l \int_{-1}^{1} P_l(x) P_m(x) dx = c_m \cdot \frac{2}{2m+1}$

$\therefore c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx$

$= \frac{2m+1}{2} \int_{0}^{1} x P_m(x) dx$

So, $c_0 = \frac{1}{4}$

$c_1 = \frac{1}{2}$

$c_2 = \frac{5}{16}$

$c_3 = 0$

$c_4 = -\frac{3}{32}$

$c_5 = 0$

$\vdots$

$\therefore f(x) = \frac{1}{4}P_0(x) + \frac{1}{2}P_1(x) + \frac{5}{16}P_2(x) + \left(-\frac{3}{32}\right) \cdot P_4(x) + \cdots$

3\. a) we know that

$$H_n(x) = (-1)^n H_n(-x)$$

$$\therefore H_{2n+1}(-x) = -H_{2n+1}(x)$$

$$H_{2n}(-x) = H_{2n}(x)$$

so
$$\int_{-\infty}^{0} H_{2n+1}(x)\, e^{-x^2/2}\, dx = (-1)\int_{\infty}^{0} H_{2n+1}(-x)\, e^{-x^2/2}\, dx$$

$x \to -x$

$$= -\int_{0}^{\infty} H_{2n+1}(x)\, e^{-x^2/2}\, dx$$

$$\therefore \int_{-\infty}^{\infty} H_{2n+1}\, e^{-x^2/2}\, dx = 0$$

or, $\int_{-\infty}^{\infty} H_n e^{-x^2/2}\, dx = 0$ when n is odd

b)
$$-t^2 + 2tx - x^2/2 = -\tfrac{1}{2}(x-2t)^2 + t^2$$

$$\rightarrow \int_{-\infty}^{\infty} e^{-t^2+2tx-x^2/2}\, dx = e^{t^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-2t)^2}\, dx$$

$$= \sqrt{2\pi}\, e^{t^2} = \sqrt{2\pi}\sum_{n=0}^{\infty}\frac{t^{2n}}{n!}$$

$$\text{LHS} = \int_{\infty}^{\infty} g(x,t)\, e^{-x^2/2}\, dx = \sum_{m=0}^{\infty}\int_{-\infty}^{\infty}\frac{t^m}{m!} H_m\, e^{-x^2/2}\, dx$$

compare coeff. of t^{2n} : $m = 2n$

$$\rightarrow \frac{1}{(2n)!}\int_{-\infty}^{\infty} H_{2n}\, e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}}{n!}$$

$$\therefore \int_{-\infty}^{\infty} H_{2n}\, e^{-x^2/2}\, dx = \sqrt{2\pi}\,\frac{(2n)!}{n!}$$

or $\int_{-\infty}^{\infty} H_n e^{-x^2/2}\, dx = \frac{\sqrt{2\pi}\, n!}{(n/2)!}$ when n is even.

4. $\mathcal{L}\left[\int_0^t f(x)\,dx\right] = \int_0^\infty e^{-st}\int_0^t f(x)\,dx\,dt$

$$= -\frac{1}{s} e^{-st}\int_0^t f(x)\,dx\Big|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st} f(t)\,dt$$

$$= \frac{1}{s}\mathcal{L}[f]$$

5. See my lecture note