Department of Physics, Korea University

Mathematical Physics II, Final Exam (June 10, 2006 by E. Won)

Note: *You must show all the calculational steps explicitly. Otherwise you will receive a cut in marks.*

1. The one-dimensional diffusion with a plane source at $x = 0$ is

$$-D\frac{d^2\phi(x)}{dx^2} + K^2 D\phi(x) = Q\delta(x)$$

where $\phi(x)$ is the flux, $Q\delta(x)$ is the source, and D, K^2 are constants. Solve with Fourier analysis, as follows. First, Fourier transform each term of the equation to get an equation for $g(k)$, the Fourier transform of $\phi(x)$. Now solve for $g(k)$, which requires only algebra. Finally, use your solution for $g(k)$ to write an integral expression for ϕ (*You do not need to solve the integral*).
Note: we define the Fourier transform as

$$g(k) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} e^{ikx} f(x)\, dx.$$

2. Expand in a Legendre series ($P_n(x)$) the function $f(x)$ given by

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$

up to the order $P_n(x)$ where n=4. You may use the below equations.

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \end{aligned}$$

3. Show the following.

$$\int_{-\infty}^{\infty} H_n(x)\, e^{-\frac{x^2}{2}}\, dx = \begin{cases} \sqrt{2\pi}\, n!/(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

where $H_n(x)$ represent Hermite polynomials. (Hint: For n is odd, use the property of the parity. For n is even, start with $e^{-t^2+2tx-x^2/2}$ and integrate it from $-\infty$ to ∞.)

4. Show that

$$\mathcal{L}\left[\int_0^t f(x)\, dx\right] = \frac{1}{s}\mathcal{L}\left[f\right]$$

when $\int_0^\infty f(x)\, dx$ is finite. $\mathcal{L}[\ ]$ represents the Laplace transform operator.

5. Show that

$$\int_0^\infty e^{-x} x^k L_n^k(x) L_m^k(x)\, dx = \frac{(n+k)!}{n!}\delta_{nm}$$

where $L_n^k(x)$ represent associate Laguerre polynomials.