Ch 16 Integral Equations

16.1 Introduction

. Definitions

Fredholm eq. of the 1st Kind

$$f(x) = \int_a^b K(x,t)\phi(t)\,dt$$

Fredholm eq. of the 2nd kind

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)\,dt$$

Volterra eq. of the 1st Kind

$$f(x) = \int_a^x K(x,t)\phi(t)\,dt$$

Volterra eq. of the 2nd Kind

$$\phi(x) = f(x) + \int_a^x K(x,t)\phi(t)\,dt$$

$\phi(t)$: unknown function

$K(x,t)$: kernel

$f(x)$: assumed to be known

(Some diffusion and transport phenomena cannot be represented by differential equations)

Ex 16.1.1 Momentum Representation in Quantum Mechanics

Γ the Schrodinger eq. is

$$-\frac{\hbar^2}{2m}\nabla^2\phi(\vec{r}) + V(\vec{r})\phi(\vec{r}) = E\phi(\vec{r})$$

or $$\nabla^2\phi(\vec{r}) + a^2\phi(\vec{r}) = v(\vec{r})\phi(\vec{r})$$

where

$$a^2 = \frac{2mE}{\hbar^2} \qquad v(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})$$

We can generalize $(\nabla^2 + a^2)\phi(\vec{r}) = v(\vec{r})\phi(\vec{r})$ as

$$(\nabla^2 + a^2)\phi(\vec{r}) = \int v(\vec{r},\vec{r}\,')\phi(\vec{r}\,')\,d^3r'$$

then $\quad v(\vec{r},\vec{r}\,') = v(\vec{r}\,')\,\delta(\vec{r}-\vec{r}\,')$

reduces to the original eq. (local interaction.)

We also had the Fourier transform ψ and Ψ

$$\Psi(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\int \psi(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}}\, d^3r$$

$$\psi(\vec{r}) = \frac{1}{(2\pi)^{3/2}}\int \Psi(\vec{k})\, e^{i\vec{k}\cdot\vec{r}}\, d^3k$$

From $\quad (\nabla^2+a^2)\psi(\vec{r}) = \int v(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,')\,d^3r'$, we get

$$\int e^{-i\vec{k}\cdot\vec{r}}(\nabla^2+a^2)\psi(\vec{r})\,d^3r = \int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\int v(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,')\,d^3r'$$

Integrating LHS by parts two times

$$\int(-k^2+a^2)\psi(\vec{r})\,e^{-i\vec{k}\cdot\vec{r}}\,d^3r = (2\pi)^{3/2}(-k^2+a^2)\,\Psi(\vec{k})$$

$$= \frac{1}{(2\pi)^{3/2}}\iiint v(\vec{r},\vec{r}\,')\,\Psi(\vec{k}\,')\,e^{-i(\vec{k}\cdot\vec{r})+i\vec{k}\,'\cdot\vec{r}\,'}\,d^3r'\,d^3r\,d^3k'$$

If we take

$$f(\vec{k},\vec{k}\,') = \frac{1}{(2\pi)^{3/2}}\iint v(\vec{r},\vec{r}\,')\,e^{-i(\vec{k}\cdot\vec{r}-\vec{k}\,'\cdot\vec{r}\,')}\,d^3r'\,d^3r$$

Above eq. becomes

$$(2\pi)^{3/2}(-k^2+a^2)\,\Psi(\vec{k}) = \int f(\vec{k},\vec{k}\,')\,\Psi(\vec{k}\,')\,d^3k'$$

↳ a Fredholm eq. of 2nd kind

Note: diff eq. becomes integral eq. in Fourier transformed (momentum) space.

- Transforming of a Differential Eq. into an Integral Eq.

starting with a linear 2nd-order ODE:

$$y'' + A(x)\,y' + B(x)\,y = g(x) \qquad \text{with } y(a)=y_0 \quad y'(a)=y_0'$$

By integrating,

$$y'(x) = -\int_a^x A(t)\,y'(t)\,dt - \int_a^x B(t)\,y(t)\,dt + \int_a^x g(t)\,dt + y_0'$$

$$= -A\,y(x) + A(a)\,y(a) - \int_a^x (B - A')\,y(t)\,dt + \int_a^x g(t)\,dt + y_0'$$

Integrating one more time,

$$y(x) = -\int_a^x Ay\,dx - \int_a^x du \int_a^u \left[B(t) - A'(t)\right] y(t)\,dt$$

$$+ \int_a^x du \int_a^u g(t)\,dt + \left[A(a)\,y(a) + y_0'\right](x-a) + y_0$$

Now, we will use the formula

$$\int_a^x du \int_a^u f(t)\,dt = \int_a^x (x-t)\,f(t)\,dt$$

$\because$ Since $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$,

$\dfrac{d}{dx}$ LHS : $\quad \dfrac{d}{dx}\left[\displaystyle\int_a^x du \int_a^u f(t)\,dt\right] = \displaystyle\int_a^x f(t)\,dt$

$\dfrac{d}{dx}$ RHS : $\quad \dfrac{d}{dx}\left[\displaystyle\int_a^x (x-t)\,f(t)\,dt\right] = \dfrac{d}{dx}\left[x\displaystyle\int_a^x f(t)\,dt - \int_a^x t\,f(t)\,dt\right]$

$$= \int_a^x f(t)\,dt + x\,f(x) - x\,f(x) = \int_a^x f(t)\,dt$$

$\therefore \dfrac{d}{dx}\text{LHS} = \dfrac{d}{dx}\text{RHS} \rightarrow \text{LHS} = \text{RHS} + \text{const}$

if $x \to a$, LHS = RHS = 0 $\therefore$ const = 0

$$\therefore \int_a^x du \int_a^u f(t)\,dt = \int_a^x (x-t)\,f(t)\,dt$$

Now, by applying what we just showed above,

$$y(x) = -\int_a^x \left\{A(t) + (x-t)\left[B(t) - A'(t)\right]\right\} y(t)\,dt$$

$$+ \int_a^x (x-t)\,g(t)\,dt + \left[A(a)\,y(a) + y_0'\right](x-a) + y_0$$

If we now introduce the followings :

$$k(x,t) = (t-x)\left[B(t) - A'(t)\right] - A(t)$$

$$f(x) = \int_a^x (x-t)\,g(t)\,dt + \left[A(0)y_0 + y_0'\right](x-a) + y_0$$

then the above eq. becomes

$$y(x) = f(x) + \int_a^x k(x,t)\,y(t)\,dt$$

: Volterra eq. of the 2nd Kind

Ex 16.1.2 Linear Oscillator Eq.

┌ Consider

$$y'' + \omega^2 y = 0 \qquad \text{with} \quad y(0) = 0 \;, \; y'(0) = 1$$

This gives

$$A(x) = 0 \;, \; B(x) = \omega^2 \;, \; g(x) = 0 \quad \text{with } a = 0$$

$$\therefore \; k(x,t) = (t-x)\cdot\omega^2$$

$$f(x) = x$$

so our Volterra eq. of the 2nd Kind becomes

$$y(x) = x + \omega^2 \int_0^x (t-x)\,y(t)\,dt$$

and is equivalent to the original differential equation + initial conditions ┘

Let us consider the linear oscillator eq. but with the following boundary cond. $y(0) = 0$, $y(b) = 0$ b: upper bound

$y'(0)$ is not given → we start from scratch:

From $y'' + \omega^2 y = 0$, we integrate it once

$$y' = -\omega^2 \int_0^x y\,dt + y'(0)$$

$$y = -\omega^2 \int_0^x du \int_0^u y\,dt + y'(0)\,x$$

$$= -\omega^2 \int_0^x (x-t)\,y(t)\,dt + y'(0)\,x$$

To eliminate unknown $y'(0)$, we impose $y(b)=0$ to the above

$$\omega^2 \int_0^b (b-t)\, y(t)\, dt = b y'(0) \quad \text{and it gives}$$

$$y(x) = -\omega^2 \int_0^x (x-t)\, y(t)\, dt + \frac{x}{b} \cdot \omega^2 \int_0^b (b-t)\, y(t)\, dt$$

then, $\frac{x}{b}(b-t) - (x-t) = x - \frac{xt}{b} - x + t = \frac{t}{b}(b-x)$

$$y(x) = -\omega^2 \int_0^x (x-t)\, y(t)\, dt + \omega^2 \frac{x}{b} \int_0^x (b-t)\, y(t)\, dt + \omega^2 \frac{x}{b} \int_x^b (b-t)\, y(t)\, dt$$

$$= -\omega^2 \int_0^x (x-t)\, y(t)\, dt + \omega^2 \int_0^x \left[(x-t) + \frac{t}{b}(b-x) \right] y(t)\, dt$$

$$+ \omega^2 \cdot \frac{x}{b} \int_x^b (b-t)\, y(t)\, d\tau$$

$$= \omega^2 \int_0^x \frac{t}{b}(b-x)\, y(t)\, dt + \omega^2 \int_x^b \frac{x}{b}(b-t)\, y(t)\, dt$$

So, if we define our kernel to be

$$k(x,t) = \begin{cases} t/b\,(b-x) & t < x \\ x/b\,(b-t) & t > x \end{cases}$$

we arrive

$$y(x) = \omega^2 \int_0^b k(x,t)\, y(t)\, dt$$

: a homogeneous Fredholm eq. of 2nd kind.

Now, our new kernel $k(x,t)$, has following properties

1. symmetric : $k(x,t) = k(t,x)$
2. It is continuous

$$\left. \frac{t}{b}(b-x) \right|_{t=x} = \left. \frac{x}{b}(b-t) \right|_{t=x}$$

3. $\partial k(x,t)/\partial t$: discontinuous

16.2 Integral Transforms, Generating Functions

List of integral transforms so far
we disscussed.

If $\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} \phi(t) dt$

then $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} \psi(t) dt$: Fourier

If $\psi(x) = \int_{0}^{\infty} e^{-xt} \phi(t) dt$

then $\phi(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{xt} \psi(t) dt$: Laplace

If $\psi(x) = \int_{0}^{\infty} t^{x-1} \phi(t) dt$ then

$\phi(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} x^{-t} \psi(t) dt$: Mellin (did we discuss?)

If $\psi(x) = \int_{0}^{\infty} t \phi(t) J_\nu(xt) dt$

then $\phi(x) = \int_{0}^{\infty} t \psi(t) J_\nu(xt) dt$ Hankel.

Ex 16.2.1. Fourier transform solution

Let us consider a Fredholm eq. of the 1st kind
with a kernel of the general type $k(x-t)$

$$f(x) = \int_{-\infty}^{\infty} k(x-t) \phi(t) dt$$

We apply Fourier convolution theorem: so

$$f(x) = \int_{-\infty}^{\infty} K(\omega) \Phi(\omega) e^{-i\omega x} d\omega$$

with $K(\omega)$, $\Phi(\omega)$, and $F(\omega)$ are Fourier
transforms of $k(x)$, $\phi(x)$, and $f(x)$, respectively.

Now, from $\quad f(x) = \int_{-\infty}^{\infty} K(\omega') \Phi(\omega') e^{-i\omega' x} d\omega,$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\omega')\Phi(\omega')\, e^{-i\omega' x} d\omega\; e^{i\omega x} dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\omega)\Phi(\omega)\, e^{ix(\omega-\omega')} dx\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, K(\omega)\Phi(\omega)\, \delta(\omega-\omega')\, d\omega'$$

$$= K(\omega)\Phi(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} F(\omega) \qquad \left(\because \frac{1}{\sqrt{2\pi}}\int_{-\infty} f(x)\, e^{i\omega x} dx = F(\omega)\right)$$

Then we can say

$$\Phi(\omega) = \frac{1}{\sqrt{2\pi}} \cdot \frac{F(\omega)}{K(\omega)},$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \Phi(\omega)\, e^{-i\omega x} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(\omega)}{K(\omega)}\, e^{-i\omega x} d\omega$$

Ex 16.22 Generalized Abel eq., convolution Th.

The generalized Abel eq. is

$$f(x) = \int_0^x \frac{\phi(t)}{(x-t)^\alpha} dt \qquad 0<\alpha<1 \qquad \text{with} \quad \begin{array}{l} f(x): \text{known} \\ \phi(x): \text{unknown} \end{array}$$

Taking the Laplace transform of both sides

$$\mathcal{L}\{f(x)\} = \mathcal{L}\left\{\int_0^x \frac{\phi(t)}{(x-t)^\alpha} dt\right\}$$

$$= \mathcal{L}\{x^{-\alpha}\}\, \mathcal{L}\{\phi(x)\}$$

$\because$ Laplace convolution th. $\quad f_1(s) = \mathcal{L}\{F_1(t)\}, \quad f_2(s) = \mathcal{L}\{F_2(t)\}$

$$f_1(s)\, f_2(s) = \mathcal{L}\left\{\int_0^t F_1(t-z)\, F_2(z)\, dz\right\}$$

Since we know that

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad \mathcal{L}\{x^{-\alpha}\} = \frac{(-\alpha)!}{s^{-\alpha+1}}$$

So, $$\mathcal{L}\{\phi(x)\} = \frac{1}{\mathcal{L}\{x^{-\alpha}\}} \mathcal{L}\{f(x)\}$$

$$= \frac{s^{1-\alpha}}{(-\alpha)!} \mathcal{L}\{f(x)\}$$

$$\because \mathcal{L}\{x^{\alpha-1}\} = \frac{(\alpha-1)!}{s^{-\alpha}}$$

Dividing by s,

$$\frac{1}{s}\mathcal{L}\{\phi(x)\} = \frac{s^{-\alpha}}{(-\alpha)!}\mathcal{L}\{f(x)\} = \frac{\mathcal{L}\{x^{\alpha-1}\}\mathcal{L}\{f(x)\}}{(-\alpha)!\,(\alpha-1)!}$$

[Note that $(-\alpha)!\,(\alpha)! = \dfrac{\pi\alpha}{\sin\pi\alpha}$,

and the Laplace convolution theorem gives

$$\frac{1}{s}\mathcal{L}\{\phi(x)\} = \mathcal{L}\{1\}\mathcal{L}\{\phi(x)\} = \mathcal{L}\left\{\int_0^x \phi(t)\,dt\right\} \Big]$$

So, we get

$$\frac{1}{s}\mathcal{L}\{\phi(x)\} = \frac{\alpha}{1}\cdot\frac{\sin\pi\alpha}{\pi\alpha}\mathcal{L}\left\{\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt\right\}$$

$$= \frac{\sin\pi\alpha}{\pi}\mathcal{L}\left\{\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt\right\}$$

So, $$\int_0^x \phi(t)\,dt = \frac{\sin\pi\alpha}{\pi}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt$$

and finally by differentiating

$$\phi(x) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt \Big]$$

- Generating Functions

Suppose we have

$$f(x) = \int_{-1}^{1} \frac{\phi(t)}{(1-2xt+t^2)^{1/2}}\,dt \qquad -1 \le x \le 1$$

Note that $(1-2xt+t^2)^{-1/2} = \sum_{r=0}^{\infty} P_r(t)\,x^r$

and assume $$\phi(t) = \sum_{n=0}^{\infty} a_n P_n(t)$$ $P_n(t)$: Legendre polynomials

Then $f(x) = \int_{-1}^{1} \sum_{n=0}^{\infty} a_n P_n(t) \sum_{r=0}^{\infty} P_r(t) x^r \, dt$

since $\int_{-1}^{1} P_m(x) P_n(x) \, dx = \frac{2\delta_{mn}}{2n+1}$

$$f(x) = \sum_{r=0}^{\infty} \frac{2a_r}{2r+1} x^r$$

a_n can be identified by differentiating n times and let $x=0$

$$\rightarrow f^{(n)}(0) = n! \frac{2}{2n+1} a_n$$

$$\therefore \phi(t) = \sum_{n=0}^{\infty} a_n P_n(t)$$

$$= \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{f^{(n)}(0)}{n!} P_n(t)$$

16.3. Neumann Series, Separable (degenerate) kernels

- Neumann Series

Suppose we are given with

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t) \phi(t) \, dt$$

: Fredholm eq. of the 2nd kind.

(we want to find $\phi(x)$!)

Let's try $\phi(x) \simeq \phi_0(x) = f(x)$

(Saying λ or $\int_a^b K(x,t)\phi(t)dt$ are small)

To improve this, we may try

$$\phi_1(x) = f(x) + \lambda \int_a^b K(x,t) f(t) \, dt$$

repeating this

$$\phi_2(x) = f(x) + \lambda \int_a^b K(x,t_1) f(t_1) \, dt_1$$

$$+ \lambda^2 \int_a^b \int_a^b K(x,t_1) K(t_1,t_2) f(t_2) \, dt_2 \, dt_1$$

and $\phi_n(x) = \sum_{i=0}^{n} \lambda^i u_i(x)$

where $u_0(x) = f(x)$

$$u_1(x) = \int_a^b K(x,t_1) f(t_1)\, dt_1$$

$$u_2(x) = \int_a^b \int_a^b K(x,t_1)\, K(t_1, t_2)\, f(t_2)\, dt_2 dt_1$$

$\vdots$

$$u_n(x) = \int\int \cdots \int K(x,t_1) K(t_1 t_2) \cdots K(t_{n-1}, t_n)\, f(t_n) dt_n \cdots dt_1$$

We expect that our solution $\phi(x)$ will be

$$\phi(x) = \lim_{n\to\infty} \phi_n(x) = \lim_{n\to\infty} \sum_{i=0}^{n} \lambda^i u_i(x)$$

provided that the infinite series converges.

Note that $\left|\lambda^n u_n(x)\right| \le |\lambda^n| \cdot |f|_{max} \cdot |K|_{max}^n \cdot |b-a|^n$

where $|f|_{max}$: max. value of $|f(x)|$ in $[a,b]$

$|K|_{max}$ " $|K(x,t)|$ in the x-t plane

(Cauchy's ratio test: if $\frac{a_{n+1}}{a_n} \le r < 1 \ni \forall$ sufficiently large n and r is independent of n $\sum a_n$ is convergent)

So, we have convergence if

$$|\lambda|\, |K|_{max} \cdot |b-a| < 1.$$

If not satisfied, we may or may not have convergence.

Now, we write

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\, \phi(t)\, dt \quad \text{as}$$

$$\phi = \lambda K \phi + f, \quad \text{with } K \text{ represent } \int_a^b K(x,t)[\ ]\, dt$$

solving for ϕ, $\phi = \dfrac{f}{(1-\lambda K)}$

and binomial expansion leads to above series expansion.

The convergence of Neumann series

$\Updownarrow$

$(1-\lambda K)^{-1}$ existance

Ex 16.3.1 Neumann Series Solution

Consider

$$\phi(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\,\phi(t)\,dt.$$

We start with

$$\phi_0(x) = x$$

then, $\phi_1(x) = x + \frac{1}{2}\int_{-1}^{1} (t-x)\,t\,dt = x + \frac{1}{2}\left(\frac{1}{3}t^3 - \frac{1}{2}t^2 x\right)\Big|_{-1}^{1}$

$$= x + \frac{2}{6} = x + \frac{1}{3}$$

$\phi_2 = f + \lambda\int K f\,dt + \lambda^2 \int\int K K f\,dt\,dt$

$= f + \lambda\int K\left(\int (f + Kf)\,dt\right)dt$

$= f + \lambda \int K \phi_1\,dt$

$$\phi_2(x) = x + \frac{1}{2}\int_{-1}^{1}(t-x)\,t\,dt + \frac{1}{2}\int_{-1}^{1}(t-x)\frac{1}{3}dt$$

$$= x + \frac{1}{3} + \frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2}t^2 - xt\right)\Big|_{-1}^{1}$$

$$= x + \frac{1}{3} - \frac{x}{3}$$

$$\phi_3(x) = x + \frac{1}{2}\int_{-1}^{1}(t-x)\left(x + \frac{1}{3} - \frac{x}{3}\right)dt$$

$$= x + \frac{1}{3} - \frac{x}{3} + \left(\frac{-1}{2}\right)\int_{-1}^{1}(t-x)\frac{x}{3}\,d$$

$$= x + \frac{1}{3} - \frac{x}{3} + \left(\frac{-1}{2}\right)\left(\frac{t^2}{6} - \frac{x^2}{3}t\right)\Big|_{-1}^{1}$$

$$= x + \frac{1}{3} - \frac{x}{3} - \frac{x^2}{3}$$

and by induction, we can say that

$$\phi_{2n}(x) = x + \sum_{s=1}^{n} (-1)^{s-1} 3^{-s} - x\sum_{s=1}^{n}(-1)^{s-1} 3^{-s}$$

letting $n \to \infty$

$$\phi(x) = \lim_{n\to\infty}\phi_{2n}(x) = x + \lim_{n\to\infty}\sum_{s=1}^{n}\frac{(-1)^{s-1}}{3^s} - x\lim_{n\to\infty}\frac{(-1)^{s-1}}{3^{-s}}$$

Now, $\lim_{n\to\infty}\sum_{s=1}^{n}\frac{(-1)^{s-1}}{3^s} = \frac{1}{3} - \frac{1}{3^2} + \frac{1}{3^3} - \frac{1}{3^4} + \frac{1}{3^5} - \frac{1}{3^6} + \cdots$

$$= \frac{1}{3}\left(1-\frac{1}{3}\right) + \frac{1}{3^3}\left(1-\frac{1}{3}\right) + \frac{1}{3^5}\left(1-\frac{1}{3}\right) + \cdots$$

$$= \frac{2}{3^2}\left[1 + \frac{1}{3^2} + \frac{1}{3^4} + \cdots\right]$$

$$= \frac{2}{3^2}\left[\frac{1}{1-\frac{1}{3^2}}\right] = \frac{2}{3^2}\cdot\frac{3^2}{8} = \frac{1}{4}$$

$$\phi(x) = \frac{3}{4}x + \frac{1}{4} \rfloor$$

- Separable Kernel

Suppose our kernel is separable, in the sense that

$$K(x,t) = \sum_{j=1}^{n} M_j(x) N_j(t)$$

where n, the upper limit is finite.

Then the Fredholm eq. of the 2nd kind yields

$$\phi(x) = f(x) + \lambda \sum_{j=1}^{n} M_j(x) \int_a^b N_j(t)\phi(t)\,dt$$

Since the integral w.r.t t is a constant,

$$\int_a^b N_j(t)\phi(t)\,dt = c_j$$

So $\quad \phi(x) = f(x) + \lambda \sum_{j=1}^{n} c_j M_j(x)$

(we still have to find c_j)

Now, $\quad c_i = \int_a^b N_i(t)\phi(t)\,dt$

$$= \int_a^b N_i(x) \left[f(x) + \lambda \sum_{j=1}^{n} M_j(x) c_j \right] dx$$

$$= \int_a^b N_i(x) f(x)\,dx + \lambda \sum_{j=1}^{n} \int_a^b N_i(x) M_j(x)\,dx \cdot c_j$$

$$= b_i + \lambda \sum_{j=1}^{n} a_{ij} c_j$$

with $\quad b_i = \int_a^b N_i(x) f(x)\,dx, \quad a_{ij} = \int_a^b N_i(x) M_j(x)\,dx$

In matrix form, $\quad (A = (a_{ij}))$

$c_i = b_i + \lambda \sum_{j=1}^{n} a_{ij} c_j$ becomes $\vec{b} = \vec{c} - \lambda A \vec{c}$

$\rightarrow \vec{b} = (1 - \lambda A)\vec{c}$ or $\vec{c} = (1-\lambda A)^{-1} \vec{b}$

If $f(x) = 0$, then $\vec{b} = 0$. $\therefore$ to get the solution $|1 - \lambda A| = 0$

Ex 16.3.2

Consider the following homogeneous Fredholm eq.

$$\phi(x) = \lambda \int_{-1}^{1} (t+x)\phi(t)\,dt$$

so, $M_1 = 1$ $\quad M_2 = x$

$N_1 = t$ $\quad N_2 = 1$

Since $a_{ij} = \int_a^b N_i(x) M_j(x)\,dx$

$$a_{11} = \int_{-1}^{1} x\,dx = 0 \quad , \quad a_{12} = \int_{-1}^{1} x^2\,dx = \frac{x^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$a_{21} = \int_{-1}^{1} 1\,dx = 2 \qquad a_{22} = \int_{-1}^{1} x\,dx = 0$$

$b_1 = b_2 = 0$

From $|1 - \lambda A| = 0$, $\begin{vmatrix} 1 & -\frac{2}{3}\lambda \\ -2\lambda & 1 \end{vmatrix} = 0 \rightarrow 1 - \frac{4}{3}\lambda^2 = 0$

$\therefore \lambda = \pm \frac{\sqrt{3}}{2}$

From $\vec{b} = \vec{c} - \lambda A \vec{c}$, $\vec{b} = 0$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \pm \frac{\sqrt{3}}{2} \begin{pmatrix} 0 & \frac{2}{3} \\ 2 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$c_1 = \pm \frac{1}{\sqrt{3}} c_2$$

Finally, with a choice of $c_1 = 1$,

$$\phi(x) = f(x) + \lambda \sum_{j=1}^{n} c_j M_j(x)$$

$\lambda = \frac{\sqrt{3}}{2}$: $\phi_1(x) = \frac{\sqrt{3}}{2}\left(1 + \sqrt{3}x\right)$

$\lambda = -\frac{\sqrt{3}}{2}$: $\phi_2(x) = -\frac{\sqrt{3}}{2}\left(1 - \sqrt{3}x\right)$