ch14 Fourier Series

14.1 General Properties

A Fourier series is defined as an expansion of a function $f(x)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

From the orthogonality of $\cos nx$ and $\sin nx$, we have

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx \qquad n = 0, 1, 2, \cdots$$

subject to the requirement that the integrals exist.

: $f(x)$ have only a finite number of finite discontinuities in $[0, 2\pi]$

In exponential form

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad \text{in which}$$

$$c_n = \frac{1}{2}(a_n - ib_n) \qquad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

and $c_0 = \frac{1}{2} a_0$

Complex Variables — Abel's Theorem

consider $f(z) = \sum_{n=0}^{\infty} C_n z^n = \sum_{n=0}^{\infty} C_n r^n e^{in\theta}$

separating real & imaginary parts we get

$$u(r,\theta) = \sum_{n=0}^{\infty} C_n r^n \cos n\theta \qquad v(r,\theta) = \sum_{n=0}^{\infty} C_n r^n \sin n\theta$$

Abel's theorem: if $u(1,\theta)$ and $v(1,\theta)$ are convergent for a given θ, then

$$u(1,\theta) + iv(1,\theta) = \lim_{r \to 1} f(re^{i\theta})$$

(will not prove it)

Ex 14.1.1)

Consider $\sum_{n=1}^{\infty} \frac{1}{n} \cos nx$, $x \in (0, 2\pi)$

Consider also $\sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{r^n e^{inx}}{n} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{r^n e^{-inx}}{n}$

Since $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ (p355, Maclaurin expansion)

we get $-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

$$\therefore \sum_{n=1}^{\infty} \frac{r^n \cos nx}{n} = -\frac{1}{2}\left[\ln(1-re^{ix}) + \ln(1-re^{-ix})\right]$$

$$= -\frac{1}{2} \ln(1-re^{ix})(1-re^{-ix})$$

$$= -\frac{1}{2} \ln\left(1+r^2-2r\cos x\right) = -\ln\left[(1+r^2)-2r\cos x\right]^{1/2}$$

letting $r=1$ and using Abel's theorem, we see that

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n} = -\ln\left(2-2\cos x\right)^{1/2} = -\ln\left(2\sin\frac{x}{2}\right) \quad x \in (0, 2\pi)$$

$\left(1-\cos x = \frac{1}{2}\sin^2\frac{x}{2}\right)$

· Completeness

Assume $f(z)$ is analytic. Then we expand $f(z)$ in a Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} d_n z^n .$$

On the unit circle $z = e^{i\theta}$, $f(z)$ becomes

$$f(z) = f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta}$$

which becomes the complex Fourier series.

(now $f(z)$ can be written in terms of the complex Fourier series)

: this is limited arguement as $z = e^{i\theta}$ has to be satisfied.

We also have

$$\int_0^{2\pi} \sin mx \sin nx \, dx = \begin{cases} \pi \, \delta_{mn} & m \neq n \\ 0 & m = 0 \end{cases}$$

$$\int_0^{2\pi} \cos mx \cos nx \, dx = \begin{cases} \pi \, \delta_{mn} & m \neq n \\ 2\pi & m = n = 0 \end{cases}$$

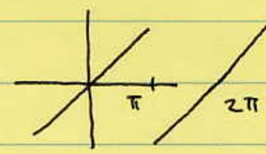
$$\int_0^{2\pi} \sin mx \cos nx \, dx = 0 \quad \ni \forall \ m \text{ and } n$$

For the complex eigenfunctions,

$$\int_0^{2\pi} (e^{imx})^* e^{inx} \, dx = 2\pi \, \delta_{mn}$$

Ex 14.1.2 Sawtooth wave

⌈ consider $f(x) = \begin{cases} x & 0 \leq x < \pi \\ x - 2\pi & \pi < x \leq 2\pi \end{cases}$



change our integral to $[-\pi, \pi]$

$$b_n = \int_{-\pi}^{\pi} x \sin nx \, dx = x \cdot \left(-\frac{1}{n}\right) \cos nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \left(-\frac{1}{n}\right) \cos nx \, dx$$

$$= -\frac{2\pi}{n} \cos n\pi + 2 \cdot \frac{1}{n^2} \sin n\pi$$

$$= -\frac{2\pi}{n} (-1)^n = \frac{2\pi}{n} (-1)^{n+1}$$

$$\therefore \ f(x) = x = 2 \left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots (-1)^{n+1} \frac{\sin nx}{n} + \cdots \right]$$ ⌋

14.2 Advantages, Uses of Fourier Series

- Discontinuous functions
  - : Fourier series can represent discontinuous functions
- Periodic functions
  - : in interval $[-\pi, \pi]$ $\quad \sin nx$ is odd
  - $\cos nx$ " even
- change of interval

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right]$$

how much useful is this?

with $a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} dt$, $n = 0, 1, 2, 3 \cdots$ $\quad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} dt$, $n = 1, 2, 3 \cdots$

14.3 Application of Fourier Series

Ex 14.3.1 $\ulcorner$ $f(x) = 0 \qquad -\pi < x < 0$

$\qquad\qquad = h \qquad 0 < x < \pi$

$$a_0 = \frac{1}{\pi}\int_0^\pi h\, dt = h$$

$$a_n = \frac{1}{\pi}\int_0^\pi h \cos nt\, dt = 0 \qquad n = 1,2,3\cdots$$

$$b_n = \frac{1}{\pi}\int_0^\pi h \sin nt\, dt = \frac{h}{n\pi}(1-\cos n\pi) = \frac{2h}{n\pi} \quad \text{if } n\text{: odd}$$

$$= 0 \quad \text{" } n\text{: even}$$

$$\therefore\ f(x) = \frac{h}{2} + \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \lrcorner$$

Exam. 14.3.3 $\ulcorner$ Infinite series, Riemann Zeta Function

Consider $f(x) = x^2 \qquad -\pi < x < \pi$

$f(x)$ is even $\rightarrow b_n = 0 \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\frac{1}{3}x^3\Big|_{-\pi}^{\pi} = \frac{2}{3}\pi^2$

$$a_n = \frac{2}{\pi}\int_0^\pi x^2 \cos nx\, dx = \frac{2}{\pi}(-1)^n \frac{2\pi}{n^2} = (-1)^n \frac{4}{n^2}$$

$$\therefore\ x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos n\pi}{n^2}$$

If we set $x = \pi$, $\qquad \cos n\pi = (-1)^n$

$$\rightarrow \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\text{or} \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \equiv \zeta(2) \lrcorner$$

14.4 Properties of Fourier Series

- Convergence

If a) $f(x)$ is continuous, $-\pi \le x \le \pi$

b) $f(-\pi) = f(+\pi)$

c) $f'(x)$ is sectionally continuous

$\rightarrow$ the Fourier series for $f(x)$ will converge uniformly

(without proof)

· Integration

From $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

term-by-term integration of the series yields

$$\int_{x_0}^{x} f(x)\,dx = \frac{a_0 x}{2}\Big|_{x_0}^{x} + \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx \Big|_{x_0}^{x} - \sum_{n=1}^{\infty} \frac{b_n}{n} \cos nx \Big|_{x_0}^{x}$$

→ this results in more rapid convergence than before.

· Differentiation

The situation regarding differentiation is quite different.

For $f(x) = x, \quad -\pi < x < \pi,$

$$x = 2\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} \qquad (\text{Ex. } 14.3.2) \qquad -\pi < x < \pi$$

Differentiating term by term, we get

$$1 = 2\sum_{n=1}^{\infty} (-1)^{n+1} \cos nx$$

but RHS is not convergent. So the convergence is not garanteed.

14.5. Gibbs Phenomenon

· Summation of Series

From $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\cos nx \int_{-\pi}^{\pi} f(t) \cos nt\,dt + \sin nx \int_{-\pi}^{\pi} f(t) \sin nt\,dt \right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(t) \cos n(t-x)\,dt$$

so, $a_n \cos nx + b_n \sin nx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t) \cos n(t-x)\,dt$

Then the rth partial sum becomes

$$S_r(x) = \frac{a_0}{2} + \sum_{n=1}^{r}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt + \sum_{n=1}^{r}\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(t-x)\,dt$$

$$= \mathrm{Re}\,\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{r} e^{-i(t-x)n}\right]dt$$

Now, Let $\Sigma_r = \frac{1}{2} + \sum_{n=1}^{r} e^{-i(t-x)n}$

$$= \frac{1}{2} + e^{-i(t-x)} + e^{-i(t-x)\cdot 2} + \cdots + e^{-i(t-x)\cdot r}$$

then $e^{-i(t-x)}\Sigma_r = \frac{1}{2}e^{-i(t-x)} + e^{-i(t-x)\cdot 2} + \cdots + e^{-i(t-x)\cdot r} + e^{-i(t-x)(r+1)}$

So, $(1 - e^{-i(t-x)})\,\Sigma_r = \frac{1}{2} + \frac{1}{2}e^{-i(t-x)} - e^{-i(t-x)(r+1)}$

$$\Sigma_r = \frac{\frac{1}{2} + \frac{1}{2}e^{-i(t-x)} - e^{-i(t-x)(r+1)}}{1 - e^{-i(t-x)}}$$

$$= \frac{\frac{1}{2}e^{i/2(t-x)} + \frac{1}{2}e^{-\frac{i}{2}(t-x)} - e^{-i(t-x)(r+1) - \frac{i}{2}(t-x)}}{e^{i/2(t-x)} - e^{-i/2(t-x)}}$$

$$= \frac{2\cos(t-x)/2 - e^{-i(t-x)(r+\frac{1}{2})}}{2i\sin\frac{1}{2}(t-x)}$$

$$\mathrm{Re}(\Sigma_r) = \frac{1}{2}\cdot\frac{\sin(t-x)(r+\frac{1}{2})}{\sin\frac{1}{2}(t-x)}$$

$$\therefore\ S_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}\,dt$$

The factor $\dfrac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}$

is called Dirichlet Kernel and is shown in Sec. 1.15 as a Dirac delta distribution

- Square wave

we consider the periodic square wave

$$f(x) = \begin{cases} h/2 & 0 < x < \pi \\ -h/2 & -\pi < x < 0 \end{cases}$$

The Fourier series solution was

$$f(x) = \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots \right)$$

Since $s_r(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, \frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}\, dt$

we get

$$S_r(x) = \frac{h}{4\pi}\int_0^{\pi} \frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt - \frac{h}{4\pi}\int_{-\pi}^{0} \frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt$$

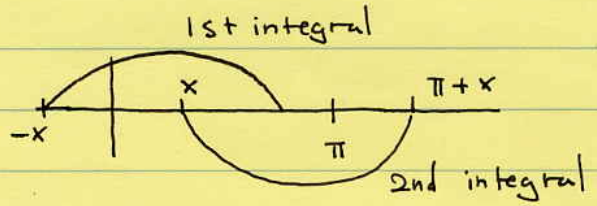
$t \to -t$ for the 2nd term

$$= \frac{h}{4\pi}\int_0^{\pi} \frac{\sin[(r+\frac{1}{2})(t-x)]}{\sin\frac{1}{2}(t-x)}dt - \frac{h}{4\pi}\int_0^{\pi} \frac{\sin[(r+\frac{1}{2})(t+x)]}{\sin\frac{1}{2}(t+x)}dt$$

$t - x = s$ in 1st, $t + x = s$ " 2nd

$$= \frac{h}{4\pi}\int_{-x}^{\pi-x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds - \frac{h}{4\pi}\int_{x}^{\pi+x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds$$

: looking at the intervals of two integrals



and the common intervals cancel each other

$$\therefore\ S_r(x) = \frac{h}{4\pi}\int_{-x}^{x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds - \frac{h}{4\pi}\int_{\pi-x}^{\pi+x} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s}ds$$

Let's take a look at the behavior at $s \to 0$ and $s \to \pi$

$$\lim_{s\to 0} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s} = r \quad \text{and} \quad \lim_{s\to\pi} \frac{\sin(r+\frac{1}{2})s}{\sin\frac{1}{2}s} = \frac{\lim_{s\to\pi}\sin(r+\frac{1}{2})s}{1} = (-1)^r$$

So we expect the contribution from the 1st integral dominate as $x \to 0$ $(r \gg 1)$

Using $r + \frac{1}{2} = p$ and $ps = \xi$

$$S_r(x) \simeq \frac{h}{4\pi}\int_{-px}^{px} \frac{\sin\xi}{\sin(\xi/2p)}\frac{d\xi}{p} = \frac{h}{2\pi}\int_0^{px} \frac{\sin\xi}{\sin(\xi/2p)}\frac{d\xi}{p}$$

· Calculation of overshoot

our partial sum was $S_r(x) = \frac{h}{2\pi}\int_0^{px} \frac{\sin\xi}{\sin(\xi/2p)} \frac{d\xi}{p}$

$S_r(0) = 0$

$S_r(x)$ increases until $\xi = ps = \pi$ ($\sin(\xi/2p)$ is still positive)

$\therefore$ The location of the overshoot maximum is at $\xi = \pi = px$

$$x = \frac{\pi}{p} \sim \frac{\pi}{r}$$

$$S_r(x)_{max} = \frac{h}{2\pi} \cdot \int_0^{\pi} \frac{\sin\xi \, d\xi}{\sin(\xi/2p)p} \cong \frac{h}{2\pi}\int_0^{\pi} \frac{\sin\xi}{\xi/2} d\xi$$

$$= \frac{h}{\pi}\int_0^{\pi} \frac{\sin\xi}{\xi} d\xi$$

(Contour integral)

$\left(\because \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ by Ex 7.1.4} \quad Si(x) = -\int_x^{\infty} \frac{\sin x}{x} dx \right)$

$$= \frac{h}{\pi}\left[\frac{\pi}{2} + Si(\pi) \right]$$

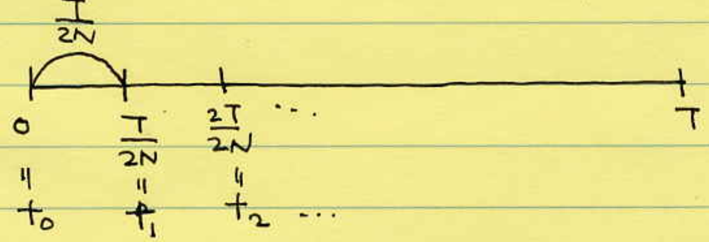
$$= \frac{h}{2} + \underbrace{\frac{h}{\pi} Si(\pi)}_{\text{overshoot}}$$

14.6 Discrete Fourier Transform

· Orthogonality over Discrete points

consider a set of 2N time values

$t_k = 0, \frac{T}{2N}, \frac{2T}{2N}, \ldots, \frac{(2N-1)T}{2N}$ for the time interval $(0, T)$



and think discrete exponential functions

$$e^{2\pi i p t_k/T}, \quad e^{2\pi i q t_k/T}$$

where p and q are integers.

Then, we can think of an orthogonality relation for these discrete exponential functions:

$$\sum_{k=0}^{2N-1} \left[e^{2\pi i p t_k/T} \right]^* e^{2\pi i q t_k/T} = ?$$

$$\sum_{k=0}^{2N-1} \left[e^{2\pi i p t_k/T} \right]^* e^{2\pi i q t_k/T} = \sum_{k=0}^{2N-1} e^{2\pi i (q-p) t_k/T}$$

$$s = q-p \quad = \sum_{k=0}^{2N-1} e^{2\pi i s t_k/T} = \sum_{k=0}^{2N-1} e^{\frac{2\pi i s k}{2N}} \qquad \left(\because t_k = \frac{kT}{2N}\right)$$

let's assume $r = \exp\left(\frac{2\pi i s}{2N}\right)$ then

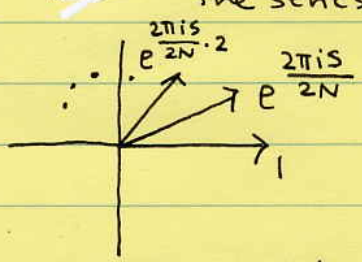
$$\sum_{k=0}^{2N-1} e^{\frac{2\pi i s k}{2N}} = \sum_{k=0}^{2N-1} r^k = 1 + r + r^2 + \cdots + r^{2N-1}$$

if $r = 1$, $\displaystyle\sum_{k=0}^{2N-1} = 1 + 1^2 + \cdots \; 1^{2N-1} = 2N$

if $r \neq 1$.

$r = e^{\frac{2\pi i s}{2N}}$ and $\displaystyle\sum_{k=0}^{2N-1} e^{\frac{2\pi i s k}{2N}} = 1 + e^{\frac{2\pi i s}{2N}} + e^{\frac{2\pi i s}{2N}\cdot 2} + \cdots + e^{\frac{2\pi i s}{2N}(2N-1)}$

and if we think the series as vector sum in 2-dim complex plane



then the total sum will be zero.

$$\therefore \sum_{k=0}^{2N-1} e^{\frac{2\pi i s t_k}{T}} = \begin{cases} 0 & \text{if } r \neq 1 \\ 2N & \text{" } r = 1 \end{cases}$$

(✻ partial sum $= \dfrac{1 - r^{2N}}{1-r}$ and $r^{2N} = e^{2\pi i s} = 1$

$\therefore$ partial sum $= 1$ if $r \neq 1$)

Discrete Fourier Transform

We introduce ω-space (or angular frequency),

$$\omega_p = \frac{2\pi p}{T} \qquad p = 0, 1, 2, \cdots 2N-1$$

Then $e^{\pm 2\pi i p t_k / T}$ becomes $e^{\pm i\omega_p t_k}$.

Consider

$$F(\omega_p) = \frac{1}{2N} \sum_{k=0}^{2N-1} f(t_k)\, e^{i\omega_p t_k}$$

Employing the orthogonality relation

$$\frac{1}{2N} \sum_{p=0}^{2N-1} \left(e^{i\omega_p t_m}\right)^* e^{i\omega_p t_k} = \delta_{mk}$$

So, $f(t_k)$ becomes

$$f(t_k) = \sum_{p=0}^{2N-1} F(\omega_p)\, e^{-i\omega_p t_k}$$

$\therefore$ $f(t_k)$ and $F(\omega_p)$ are discrete Fourier transforms of each other.