1. $$\int_{-\infty}^{\infty} \frac{d\theta}{dx}\, g(x)\, dx = \theta(x) g(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \theta \frac{dg}{dx}\, dx$$

$$= -\int_0^{\infty} \frac{dg}{dx}\, dx = -g(x)\Big|_0^{\infty} = g(0)$$

we assume $g(x)$ is bounded at $\pm\infty$.

$\therefore\ d\theta(x)/dx = \delta(x)$

2. $$\vec{\nabla}(\vec{v}\cdot\vec{v}) = \vec{\nabla}(v_x^2+v_y^2+v_z^2) = \frac{\partial}{\partial x}(v_x^2+v_y^2+v_z^2)\hat{x} + \frac{\partial}{\partial y}(v_x^2+v_y^2+v_z^2)\hat{y} + \frac{\partial}{\partial z}(v_x^2+v_y^2+v_z^2)\hat{z}$$

$$= \left(2v_x\frac{\partial v_x}{\partial x} + 2v_y\frac{\partial v_y}{\partial x} + 2v_z\frac{\partial v_z}{\partial x}\right)\hat{x} + \left(2v_x\frac{\partial v_x}{\partial y} + 2v_y\frac{\partial v_y}{\partial y} + 2v_z\frac{\partial v_z}{\partial y}\right)\hat{y} + \left(2v_x\frac{\partial v_x}{\partial z} + 2v_y\frac{\partial v_y}{\partial z} + 2v_z\frac{\partial v_z}{\partial z}\right)\hat{z}$$

$$2\vec{v}\cdot\vec{\nabla}\vec{v} = 2\left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)\vec{v}$$

$$= 2v_x\frac{\partial}{\partial x}(v_x\hat{x}+v_y\hat{y}+v_z\hat{z}) + 2v_y\frac{\partial}{\partial y}(v_x\hat{x}+v_y\hat{y}+v_z\hat{z}) + 2v_z\frac{\partial}{\partial z}(v_x\hat{x}+v_y\hat{y}+v_z\hat{z})$$

$$= \left(2v_x\frac{\partial v_x}{\partial x} + 2v_y\frac{\partial v_x}{\partial y} + 2v_z\frac{\partial v_x}{\partial z}\right)\hat{x} + \left(2v_x\frac{\partial v_y}{\partial x} + 2v_y\frac{\partial v_y}{\partial y} + 2v_z\frac{\partial v_y}{\partial z}\right)\hat{y} + \left(2v_x\frac{\partial v_z}{\partial x} + 2v_y\frac{\partial v_z}{\partial y} + 2v_z\frac{\partial v_z}{\partial z}\right)\hat{z}$$

$$2\vec{v}\times(\vec{\nabla}\times\vec{v}) = 2\vec{v}\times\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\hat{x} + \left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\hat{y} + \left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\hat{z}\right]$$

$$= \left[2v_y\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) - 2v_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right]\hat{x}$$

$$+ \left[2v_z\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) - 2v_x\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]\hat{y}$$

$$+ \left[2v_x\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) - 2v_y\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\right]\hat{z}$$

$$\text{RHS}\big|_x = \left(2v_x\frac{\partial v_x}{\partial x} + 2v_y\frac{\partial v_x}{\partial y} + 2v_z\frac{\partial v_x}{\partial z}\right)\hat{x} + \left[2v_y\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) - 2v_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right]\hat{x}$$

$$= 2v_x\frac{\partial v_x}{\partial x} + 2v_y\frac{\partial v_y}{\partial x} + 2v_z\frac{\partial v_z}{\partial x} = \text{LHS}\big|_x$$

Repeating for y and z comp. one can prove it

3. It is obvious from $\delta(g(x)) = \sum\limits_{\substack{g(a)=0, \\ g'(a)\neq 0}} \dfrac{\delta(x-a)}{|g'(a)|}$

4. For the spherical polar coord.

$$J = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

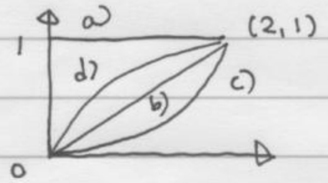
Similar way, $J = \rho$ for the cylindrical coordinates

5. $a_n = \dfrac{1}{n!}$ Now, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \dfrac{n!}{(n+1)!} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0 < 1$

$\therefore$ $\sum a_n$ is convergent according to the ratio test

6. Both diverges due to the comparison test with $\sum \frac{1}{n}$

7.



$\vec{V} = xy\,\hat{x} - y^2\,\hat{y}$ is given

$\vec{V}\cdot d\vec{r} = (xy\,\hat{x} - y^2\hat{y})\cdot(\hat{x}\,dx + \hat{y}\,dy)$

$= xy\,dx - y^2 dy$

For a), $\int \vec{V}\cdot d\vec{r}$

$$= \int_{y=0}^{1} (0\cdot y\cdot 0 - y^2 dy) + \int_{x=0}^{2} (x\cdot 1\cdot dx - 1\cdot 0)$$

$$= -\frac{1}{3} + 2 = \frac{5}{3}$$

For b) $\int \vec{V}\cdot d\vec{r} = \int_0^2 \left(x\cdot\frac{1}{2}x\cdot dx - \left(\frac{1}{2}x\right)^2\cdot\frac{1}{2}\,dx\right] = \int_0^2 \frac{3}{8}x^2\,dx = \frac{x^3}{8}\Big|_0^2 = 1$

For c) $\int \vec{V}\cdot d\vec{r} = \int_0^2 \left(x\cdot\frac{x^2}{4}\cdot dx - \frac{x^4}{16}\cdot\frac{1}{2}x\,dx\right) = \int_0^2 \left(\frac{x^3}{4} - \frac{x^5}{32}\right)dx$

$$= \frac{x^4}{16} - \frac{x^6}{192}\Big|_0^2 = \frac{2}{3}$$

For d) $\int \vec{V}\cdot d\vec{r} = \int_{t=0}^{1} (2t^3\cdot t^2\cdot 6t^2\,dt - t^4\cdot 2t\,dt) = \int_0^1 (12t^7 - 2t^5)\,dt$

$$= \frac{12}{8} - \frac{2}{6} = \frac{7}{6}$$

8 Let $A|\vec{r}_i\rangle = \lambda_i|\vec{r}_i\rangle$ and

$A|\vec{r}_j\rangle = \lambda_j|\vec{r}_j\rangle$. Then

$\langle\vec{r}_j|A|\vec{r}_i\rangle = \lambda_i\langle\vec{r}_j|\vec{r}_i\rangle$ and

$\langle\vec{r}_i|A|\vec{r}_j\rangle = \lambda_j\langle\vec{r}_i|\vec{r}_j\rangle$. and from the [illegible]nd relation

$\langle\vec{r}_j|A^{+}|\vec{r}_i\rangle = -\langle\vec{r}_j|A|\vec{r}_i\rangle = \lambda_j^{*}\langle\vec{r}_j|\vec{r}_i\rangle$

$\therefore \quad 0 = (\lambda_i + \lambda_j^{*})\langle\vec{r}_j|\vec{r}_i\rangle$

If $j=i$, we have $0 = (\lambda_i + \lambda_i^{*})\langle\vec{r}_i|\vec{r}_i\rangle$

$\therefore \quad \lambda_i + \lambda_i^{*} = 0$ in general so the eigenvalues are pure imaginary

if $j \neq i$ $\quad \langle\vec{r}_j|\vec{r}_i\rangle = 0$ so eigenvectors are orthogonal.