PYH222: Mathematical Physics Mid-Term Exam

by Prof. Eunil Won (Oct 18, 2006)

1.(10 points) If we define $\theta(x)$ to be

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

then show that

$$\frac{d\theta(x)}{dx} = \delta(x)$$

where $\delta(x)$ is the Dirac delta function.

2. (20 points) Let us *define*

$$\mathbf{v} \cdot \nabla \mathbf{F} \equiv (\mathbf{v} \cdot \nabla)\mathbf{F}$$

where $\mathbf{v}$ and $\mathbf{F}$ are arbitrary vectors. Prove that

$$\nabla(\mathbf{v} \cdot \mathbf{v}) = 2\mathbf{v} \cdot \nabla \mathbf{v} + 2\mathbf{v} \times (\nabla \times \mathbf{v}).$$

3. (10 points) Show that

$$\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$$

for a constant $a > 0$.

4. (20 points) *Jacobian* J is defined as

$$J \equiv \frac{\partial(q_1, q_2, q_3, ...)}{\partial(x_1, x_2, x_3, ...)} = \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \cdots \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}$$

where two sets $(x_1, x_2, ..., x_n)$ and $(q_1, q_2, ..., q_n)$ represent two generalized coordinate systems. Calculate the Jacobian between the Cartesian and spherical polar coordinates. Calculate the Jacobian between the Cartesian and circular cylindrical coordinates.

5. (10 points) Discuss if the following sum of series would converge or not.

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + ...$$

6. (10 points) Discuss if the following sum of series would converge or not.

$$a)\ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}, \qquad b)\ \sum_{n=2}^{\infty} \frac{1}{\log n}.$$

7. (10 points) In the $x-y$ plane, let us consider following four line segments. a) Straight lines from (0,0) to (0,1) and from (0,1) to (2,1). b) A line from (0,0) to (2,1) with $y = x/2$. c) A curve from (0,0) to(2,1) with $y = x^2/4$. d) A parametric curve of $x = 2t^3$, $y = t^2$ where $t \in [0,1]$. Now you are given with a two-dimensional vector field $\mathbf{V}(x,y) = xy\hat{\mathbf{x}} - y^2\hat{\mathbf{y}}$. Find the following line integral

$$\int \mathbf{V} \cdot d\mathbf{r}$$

for four different paths described above.

8. (10 points) Prove that the eigenvalues are pure imaginary and the eigenvectors corresponding to distinct eigenvalues are orthogonal for anti-Hermitian matrices.