1. Let us take a look at

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{2y+2\Delta y+ix+i\Delta x-2y-ix}{\Delta x+i\Delta y}$$

$$= \frac{2\Delta y+i\Delta x}{\Delta x+i\Delta y} = \frac{2\frac{\Delta y}{\Delta x}+i}{1+i\,\Delta y/\Delta x}$$

and is dependent on $\Delta y/\Delta x$ in the complex plane

$\rightarrow$ so $\frac{f(z+\Delta z)-f(z)}{\Delta z}$ cannot be defined uniquely

so is not differentiable, anywhere in the complex plane.

2. $$f(z)=\frac{1}{z(z-2)^3}=\frac{-1}{8z(1-z/2)^3}=-\frac{1}{8z}\left[1+(-3)\left(-\frac{z}{2}\right)+\frac{(-3)(-4)}{2!}\left(-\frac{z}{2}\right)^2+\frac{(-3)(-4)(-5)}{3!}\left(-\frac{z}{2}\right)^3+\cdots\right]$$

$$=-\frac{1}{8z}\left[1+\frac{3}{2}z-\frac{3}{2}z^2+\frac{5}{4}z^3+\cdots\right]$$

$$=-\frac{1}{8z}-\frac{3}{16}-\frac{3z}{16}-\frac{5}{32}z^2-\cdots$$

$\therefore$ pole at $z=0$, residue : $-\frac{1}{8}$

Now,

$$f(z)=\frac{1}{z(z-2)^3}=\frac{1}{(z-2+2)(z-2)^3}=\frac{1}{2(z-2)^3\left(1+\frac{z-2}{2}\right)}$$

$$=\frac{1}{2(z-2)^3}\left[1-\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2-\left(\frac{z-2}{2}\right)^3+\cdots\right]$$

$$=\frac{1}{2(z-2)^3}-\frac{1}{4(z-2)^2}+\frac{1}{8(z-2)}-\frac{1}{16}+\cdots$$

$\therefore$ pole at $z=2$, residue $=\frac{1}{8}$

3. Let us find out residue at $z=-a$:

$$\frac{1}{(z+a)^3 z^{1/2}} = \frac{1}{(z+a)^3}\cdot\frac{1}{(z+a-a)^{1/2}}$$

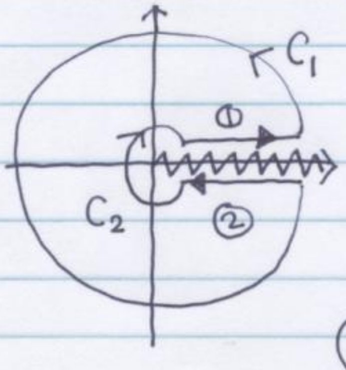
$$= \frac{1}{(z+a)^3}\cdot\frac{1}{(i)\left(1-\frac{z+a}{a}\right)^{1/2}\cdot a^{1/2}}$$

$$= \frac{1}{(i)(z+a)^3 a^{1/2}}\left[1+\left(-\frac{1}{2}\right)\left(-\frac{z+a}{a}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(-\frac{z+a}{a}\right)^2 + \cdots\right]$$

$$= \frac{1}{(i)(z+a)^3 a^{1/2}}\left[1+\frac{1}{2}\left(\frac{z+a}{a}\right)+\frac{3}{8}\cdot\frac{(z+a)^2}{a^2}+\cdots\right]$$

$\therefore$ residue : $(-i)\,\dfrac{3}{8}\cdot\dfrac{1}{a^{1/2}}\cdot\dfrac{1}{a^2} = \dfrac{-3i}{8a^{5/2}}$

Now, for the contour integral, we take



Note that there is a branch point at $z=0$:

The residue theorem gives

$$\int_{①} + \int_{C_1} + \int_{②} + \int_{C_2} = 2\pi i\left(\frac{-3i}{8a^{5/2}}\right)$$

$\int_{C_1} = 0 = \int_{C_2}$ at the limit

and $\int_{①} + \int_{②}$ becomes

$$\int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} + \int_\infty^0 \frac{dx}{(x+a)^3 x^{1/2} e^{\pi i}} = \frac{3\pi}{4a^{5/2}}$$

$$\therefore \left(1-e^{-\pi i}\right)\int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{4a^{5/2}}$$

$$\int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}} = \frac{3\pi}{8a^{5/2}}$$

4. Let us think of a unit circle with $z = e^{i\theta}$

Then, $\cos\theta = \frac{1}{2}(z + z^{-1})$, $\cos 2\theta = \frac{1}{2}(z^2 + z^{-2})$, $d\theta = (-i) z^{-1} dz$

So, our contour integral becomes

$$\oint \frac{\frac{1}{2}(z^2 + z^{-2})}{a^2 + b^2 - 2ab \cdot \frac{1}{2}(z + z^{-1})} \cdot (-i) z^{-1} \, dz$$

$$= \oint \frac{(-i)}{2} \cdot \frac{1}{z^3} \cdot \frac{(z^4+1)}{(a^2+b^2 - ab(z+z^{-1}))} \, dz$$

$$= \frac{(-i)}{2ab} \oint \frac{(z^4+1)}{(-1)\left(z^2 - \left(\frac{a^2+b^2}{ab}\right)z + 1\right) z^2} dz$$

$$= \frac{i}{2ab} \oint \frac{z^4+1}{z^2 (z - a/b)(z - b/a)} dz$$

$\therefore$ double pole at $z=0$ & a simple pole at $z = a/b$

$(\because b > a)$

at $z=0$: $$\frac{d}{dz}\left(\frac{z^4+1}{(z-a/b)(z-b/a)}\right)\Bigg|_{z=0}$$

$$= \frac{1}{(z-a/b)^2 (z-b/a)^2}\left[(z-a/b)(z-b/a) \cdot 4z^3 - (z^4+1)\left\{(z-a/b)+(z-b/a)\right\}\right]_{z=0}$$

$$= \frac{a}{b} + \frac{b}{a}$$

at $z = a/b$, residue becomes

$$\left[\left(\frac{a}{b}\right)^4 + 1\right] \Big/ \left(\frac{a}{b}\right)^2 \left(\frac{a}{b} - \frac{b}{a}\right) = \frac{a^4+b^4}{a^2 b^2 \left(\frac{a}{b} - \frac{b}{a}\right)}$$

$$= \frac{a^4+b^4}{ab(a^2-b^2)}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{a^2+b^2 - 2ab\cos\theta} d\theta = 2\pi i \cdot \frac{i}{2ab}\left[\frac{a}{b} + \frac{b}{a} + \frac{a^4+b^4}{ab(a^2-b^2)}\right]$$

5. Now, $\frac{d}{ds}\sum_{n=0}^{\infty} B_n(s)\frac{x^n}{n!} = \sum_{n=1}^{\infty}\frac{dB_n(s)}{ds}\frac{x^n}{n!}$ $\quad (\because B_0 = 1)$

$$= \frac{x^2 e^{xs}}{e^x-1} = \sum_{n=0}^{\infty} B_n(s)\frac{x^{n+1}}{n!}$$

$$= \sum_{n=1}^{\infty} B_{n-1}(s)\frac{x^n}{n!}\cdot n$$

$$\therefore \quad \frac{d}{ds}B_n(s) = n B_{n-1}(s) \qquad n = 1, 2, 3 \cdots$$

From $\frac{xe^{xs}}{e^x-1} = \sum_{n=0}^{\infty} B_n(s)\frac{x^n}{n!}$, replace $x \to -x$

then $-\frac{xe^{-sx}}{e^{-x}-1} = \sum_{n=0}^{\infty} B_n(s)\frac{(-1)^n x^n}{n!}$

Multiplying e^x on LHS (both on numerator and on denominator)

$$\frac{xe^{x(1-s)}}{e^x-1} = \sum_{n=0}^{\infty} B_n(1-s)\frac{x^n}{n!}(-1)^n$$

set $s=1$,

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n(0)\frac{(-1)^n x^n}{n!}$$

$$= \sum_{n=0}^{\infty} B_n(1)\frac{1^n}{n!}$$

$$\therefore \quad B_n(1) = (-1)^n B_n(0) \qquad n = 1, 2, 3 \cdots$$

6. $\int_0^{\infty} e^{-x^{\alpha}}dx$. Now, let $x^{\alpha} = t$, $\alpha x^{\alpha-1}dx = dt$

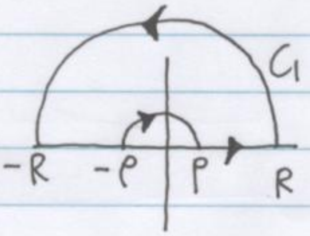
$x = t^{\frac{1}{\alpha}}$ $\quad dx = \frac{1}{\alpha}x^{1-\alpha}dt = \frac{1}{\alpha}t^{\frac{1-\alpha}{\alpha}}dt$

$$\int_0^{\infty} e^{-x^{\alpha}}dx = \int_0^{\infty} e^{-t}\left(\frac{1}{\alpha}\right)t^{\frac{1-\alpha}{\alpha}}dt$$

$$= \frac{1}{\alpha}\int_0^{\infty} e^{-t}\, t^{\frac{1-\alpha}{\alpha}+1-1}\,dt$$

$$= \frac{1}{\alpha}\int_0^{\infty} e^{-t}\, t^{\frac{1}{\alpha}-1}\,dt = \frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)$$

7. Consider $\oint \frac{\sin^2 z}{z^2}\,dz$ with the contour



so, $$\oint \frac{\sin^2 z}{z^2}\,dz = \int_{-R}^{\rho} \frac{\sin^2 x}{x^2}\,dx + \int_{\pi}^{0} \frac{\sin^2 z}{z^2}\,dz + \int_{\rho}^{R} \frac{\sin^2 x}{x^2}\,dx + \int_{C_1} \frac{\sin^2 z}{z^2}\,dz$$

For $\int_{C_1}$, having $z = Re^{i\theta}$ and let $R \to \infty$ gives zero.

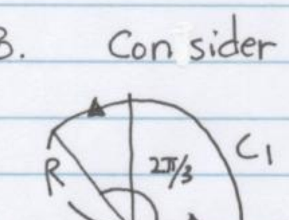
For $$\int_{\pi}^{0} \frac{\sin^2 z}{z^2}\,dz = \int_{\pi}^{0} \frac{1}{2}\frac{(1-e^{2iz})}{z^2}\,dz$$ when $\rho \ll 1$,

$$= \frac{1}{2}\int_{\pi}^{0} \frac{(-2i\rho e^{i\theta})}{\rho^2 e^{2i\theta}}\, i\rho e^{i\theta}\,d\theta$$

$$= \int_{\pi}^{0} d\theta = -\pi. \quad \text{or} \quad P\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = -\pi$$

$$\therefore \int_{0}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \left(-\frac{1}{2}\right) P\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

8. Consider the following contour



$$\oint \frac{dz}{1+z^3} = \int_{0}^{\infty} \frac{dx}{1+x^3} + \int_{\infty}^{0} \frac{e^{2\pi i/3}}{1+x^3}\,dx + \int_{C_1} \frac{dz}{1+z^3}$$

$\int_{C_1} \frac{dz}{1+z^3}$ vanishes as $R \to \infty$

simple pole at $z = e^{\pi i/3}$

$$\text{residue} = \lim_{z \to e^{\pi i/3}} \frac{(z - e^{2\pi i/3})}{z^3+1} = \frac{1}{3z^2} = \frac{1}{3e^{2\pi i/3}}$$

$$\therefore \oint \frac{dz}{1+z^3} = (1 - e^{2\pi i/3}) \int_{0}^{\infty} \frac{dx}{1+x^3} = 2\pi i \cdot \frac{1}{3e^{2\pi i/3}}$$

$$\therefore \int_{0}^{\infty} \frac{dx}{1+x^3} = \frac{2\pi i}{3} \frac{e^{-2\pi i/3}}{1 - e^{2\pi i/3}} = \frac{\pi}{3\sin\frac{\pi}{3}}$$