

PYH222: Mathematical Physics Final-Term Exam

by Prof. Eunil Won (Dec 13, 2006)

1. (10 points) Show that the function $f(z) = 2y + ix$ is not differentiable anywhere in the complex plane.

2. (20 points) Find the Laurent series of

$$f(z) = \frac{1}{z(z-2)^3}$$

about the singularities $z = 0$ and $z = 2$, separately. You should list at least four non-vanishing terms. Hence verify that $z = 0$ is a pole of order 1 and $z = 2$ is a pole of order 3, and find the residue of $f(z)$ at each pole.

3. (20 points) Evaluate

$$\int_0^\infty \frac{dx}{(x+a)^3 x^{1/2}}, \quad a > 0.$$

4. (10 points) Evaluate

$$\int_0^{2\pi} \frac{\cos 2\theta}{a^2 + b^2 - 2ab \cos \theta} d\theta, \quad b > a > 0.$$

5. (10 points) Bernoulli polynomials $B_n(s)$ are defined by

$$\frac{x e^{xs}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(s) \frac{x^n}{n!}.$$

Prove that

$$\frac{d}{ds} B_n(s) = n B_{n-1}(s), \quad n = 1, 2, 3, \dots$$

and

$$B_n(1) = (-1)^n B_n(0), \quad n = 1, 2, 3, \dots$$

6. (10 points) Express the below equation in terms of the gamma function

$$\int_0^\infty e^{-x^\alpha} dx$$

when $\alpha > 1$.

7. (10 points) Evaluate

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx.$$

8. (10 points) Evaluate

$$\int_0^\infty \frac{dx}{x^3 + 1}.$$