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Some Observable Effects of the Quantum-Mechanical Fluctuations of the Electromagnetic Field

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An intuitive explanation is given for the electromagnetic shift of energy levels by calculating the mean square amplitude of oscillation of an electron coupled to the zero-point fluctuations of the electromagnetic field. The resulting disturbance of the charge and current density of the electron gives rise to various observable effects which can be estimated in a simple classical fashion. The effects treated are the Lamb shift, the correction to the g-factor for the orbital and spin angular momenta of the electron, and the correction to the Compton scattering cross section at low energies. A simple explanation is also given for the peculiar ultraviolet divergence noticed by Pauli and Fierz in their treatment of the infra-red paradox.

I. INTRODUCTION

IT has been pointed out by Bethe[1] that the displacement of the $2S$ level of hydrogen observed by Lamb and Retherford,[2] can be simply explained as a shift in the energy of the atom arising from its interaction with the radiation field. In order to obtain this result, it is necessary to subtract from the usual infinite result an infinite energy which is essentially the electromagnetic self-energy of the electron. The residual energy shift gives the experimental result, after the introduction of a plausible cut-off in the integral over quantum energies. It is the purpose of this note to point out the existence of a simple picture of the origin of this residual finite level shift which is capable of extension to other phenomena in which virtual emission and absorption of quanta gives rise to formal divergences. In many cases it yields a simple understanding of the phenomenon and, in addition, a semiquantitative calculation of the magnitude of the effect.

Since Bethe's subtraction removes the classical electromagnetic self-energy, the remaining physically significant energy must be thought of as arising from the purely quantum-mechanical aspects of the system. The quantization of the radiation field manifests itself qualitatively through the existence of fluctuating field strengths in empty space, and it is therefore natural to inquire into the effects produced on mechanical systems by these fluctuations. The most obvious effect is the spontaneous emission of radiation from atoms in excited states. This phenomenon can be thought of as forced emission taking place under the action of the fluctuating field. Another effect which is equally simple is the existence of a fluctuation in position of a free electron. This may be thought of as a Brownian motion of the electron in equilibrium with a hohlraum, which motion persists when the temperature is reduced to absolute zero. It may be expected that this fluctuation in position will disturb the charge and current distribution arising from an electron in an atom, and hence give rise to observable effects. Weisskopf[3] has pointed out that the interaction of this fluctuation with the field which produces it gives a natural explanation for a part of the infinite self-energy of the free electron. In this paper we shall assume that the position fluctuation is a real concept, while we shall think of the energy of interaction between the field and particle fluctuations as having no physical reality. We propose to give a discussion of these effects which is as nearly intuitive as possible, postponing any attempts at formal justification to the last part of this paper.

II. THE MEAN SQUARE FLUCTUATION IN POSITION OF A FREE ELECTRON

Our starting point is the observation that the quantum-mechanical zero-point variation of the radiation field in empty space gives rise to fluctuating electric and magnetic fields whose mean square values at a point in space are given by the

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[1] H. A. Bethe, Phys. Rev. **72**, 339 (1947).
[2] W. E. Lamb, Jr. and R. C. Retherford, Phys. Rev. **72**, 241 (1947).
[3] V. F. Weisskopf, Phys. Rev. **56**, 72 (1939).

well-known relation

$$\langle E^2\rangle_{Av}=\langle B^2\rangle_{Av}=\frac{2\hbar c}{\pi}\int_0^\infty k^3dk. \tag{1}$$

In this equation the variable k refers to the wave number of a quantum, and the contribution to the mean square fluctuation arising from frequencies in the range cdk is therefore explicitly displayed. Equation (1) can be simply derived by ascribing to every normal mode of the radiation field an energy which is just the zero-point energy for an oscillator with the frequency of the normal mode. The total energy can be written either as the volume integral of the ordinary electromagnetic energy density or as a sum over normal modes, and Eq. (1) merely states the equality of these two forms.

It will now be assumed that an otherwise free electron is acted on by these fluctuating fields. The electron will be assumed to move with nonrelativistic velocities so that if $\mathbf{q}$ is its position vector, the equation of motion is

$$m(d^2\mathbf{q}/dt^2)=e\mathbf{E}. \tag{2}$$

The vector $\mathbf{E}$ is the fluctuating field specified by (1). Since Eq. (2) is linear, we can regard it as a classical equation for the quantum-mechanical expectation value of $\mathbf{q}$. For a given harmonic component of $\mathbf{E}$ the solution of (2) is obvious. We perform this integration, find the resulting value of $\langle q^2\rangle_{Av}$, and sum over frequencies using (1). We then obtain a quantity $\langle(\Delta q)^2\rangle_{Av}$, defined as the mean square fluctuation in position of a free electron

$$\langle(\Delta q)^2\rangle_{Av}=\frac{2}{\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\int_{k_0}^{\kappa}\frac{dk}{k}. \tag{3}$$

The upper and lower limits of integration are infinity and zero if the preceding assumptions are literally adhered to. The divergence at the upper limit of the integral in (3) is certainly fictitious since the equation of motion (2) neglects the spatial dependence of a given Fourier component of the electromagnetic field. This spatial dependence would give rise to a longitudinal recoil of the electron in addition to the transverse oscillation which we have included. This recoil will become important for light with a wavelength shorter than the Compton wave-length of the electron. We therefore assume that the upper limit is given by

$$\kappa=mc/\hbar. \tag{4}$$

This assumption will be further supported in the last section of this paper.

The lower limit is strictly zero, and for a free electron we therefore have the result that the mean square fluctuation in position is infinite. It should, however, be remembered that the divergence arises from very large, but very slow low frequency fluctuations. These contributions will be suppressed by the presence of any sort of binding, and the lower limit k_0 will be determined in every case from consideration of the details of the electronic motion. This point will be discussed more carefully in a later section.

III. THE LAMB SHIFT

The magnitude of this mean square fluctuation in position will be very small for any reasonable k_0, but an observable effect will arise when the electron moves in a potential with a large curvature. This can easily be seen in the following way. Consider the motion of an electron in a static field of force specified by a potential energy $V(\mathbf{q})$. The coordinates of the electron consist of a part which varies smoothly in time with the orbital frequency plus a part which fluctuates randomly in time. We designate the smooth part by $\mathbf{q}$ and the random part by $\Delta\mathbf{q}$. The instantaneous potential energy is then given by

$$V(\mathbf{q}+\Delta\mathbf{q})=[1+\Delta\mathbf{q}\cdot\boldsymbol{\nabla}+\tfrac{1}{2}(\Delta\mathbf{q}\cdot\boldsymbol{\nabla})^2+\cdots]V(\mathbf{q}). \tag{5}$$

The effective potential energy in which the particle moves will just be the average of (5) over all values of $\Delta\mathbf{q}$. Remembering that $\Delta\mathbf{q}$ has an isotropic spatial distribution, we obtain

$$\langle V(\mathbf{q}+\Delta\mathbf{q})\rangle_{Av}=[1+\tfrac{1}{6}\langle(\Delta q)^2\rangle_{Av}\nabla^2+\cdots]V(\mathbf{q}). \tag{6}$$

We thus see that the existence of the position fluctuation of the electron will effectively modify the potential in which it moves by the addition of a term proportional to the Laplacian of the potential energy. This correction will suffice if ∇^2V is sufficiently small, but in general it must be remembered that the effective potential energy is to be obtained from the original potential energy by the application of an integral operator

with a displacement kernel, which is just the distribution function for $\Delta\mathbf{q}$. This point will be amplified in a later section.

Now consider the perturbation of the levels of the hydrogen atom caused by this modification of the potential energy. We take for $V(\mathbf{q})$ the static potential of the nucleus

$$V(\mathbf{q}) = -\frac{e^2}{r}. \tag{7}$$

The expansion (6) does not converge for this potential, but for the time being it will be assumed that the first correction term in (6) is sufficient, the justification being left until later. We obtain the correction to the potential energy by combining (3), (4), and (6)

$$\Delta V(\mathbf{q}) = \frac{4e^2}{3}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2 \log\frac{mc^2}{\hbar c k_0}\delta(\mathbf{q}). \tag{8}$$

The correction to the energy of a stationary state of the atom with wave function $\psi(\mathbf{q})$ will be

$$\Delta E = \frac{4e^2}{3}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2 \log\frac{mc^2}{\hbar c k_0}|\psi(0)|^2. \tag{9}$$

This expression will be recognized as identical with the expression derived by Bethe for the level shift. The quantity hck_0 should clearly be taken equal to the average excitation energy (17.8 Rydbergs) introduced by Bethe,[1] in order to obtain approximate agreement with the experimental result of Lamb. This value for the lower cut-off occurring in the integral over quanta seems implausibly large from the arguments advanced thus far, but it will later be shown by a simple argument to be very reasonable.

It should be noted in passing that from the nature of the effect the sign of the energy shift is clear, it being obvious that the fluctuation in position must always act to weaken the effect of the potential energy.

We must now discuss the validity of the seemingly inadmissible omission of powers of the Laplacian higher than the first from the expansion (6). Suppose that the probability of finding the instantaneous fluctuation $\Delta\mathbf{q}$ in the volume element $d\Delta\mathbf{q}$ is

$$P(\Delta\mathbf{q})d\Delta\mathbf{q}. \tag{10}$$

Returning to the derivation of (6), we can write

$$\langle V(\mathbf{q}+\Delta\mathbf{q})\rangle_{\text{Av}} = \int d\Delta\mathbf{q} P(\Delta\mathbf{q})\, V(\mathbf{q}+\Delta\mathbf{q}). \tag{11}$$

We are actually interested in the integral over-all space of the product of (11) with $|\psi(\mathbf{q})|^2$, in finding an energy shift

$$\int d\mathbf{q}|\psi(\mathbf{q})|^2\langle V(\mathbf{q}+\Delta\mathbf{q})\rangle_{\text{Av}}$$

$$= \int d\mathbf{q}\int d\Delta\mathbf{q} P(\Delta\mathbf{q})\, V(\mathbf{q}+\Delta\mathbf{q})|\psi(\mathbf{q})|^2. \tag{12}$$

Taking $\Delta\mathbf{q}$ and $\mathbf{q}'=\mathbf{q}+\Delta\mathbf{q}$ as variables of integration we obtain for this average

$$\int d\mathbf{q}' V(\mathbf{q}')\int d\Delta\mathbf{q} P(\Delta\mathbf{q})|\psi(\mathbf{q}'-\Delta\mathbf{q})|^2. \tag{13}$$

We then see that by requiring only the weak condition that $|\psi(\mathbf{q})|^2$ does not have too large a curvature we can obtain for (13)

$$\int d\mathbf{q} V(\mathbf{q})\{|\psi(\mathbf{q})|^2 + \tfrac{1}{6}\langle(\Delta q)^2\rangle_{\text{Av}}\nabla^2|\psi(\mathbf{q})|^2\}$$

$$= \int d\mathbf{q}|\psi(\mathbf{q})|^2\{1+\tfrac{1}{6}\langle(\Delta q)^2\rangle_{\text{Av}}\nabla^2\} V(\mathbf{q}). \tag{14}$$

The second line is obtained by an obvious integration by parts. The second term represents the correction to the energy of the system and gives a result identical with that previously obtained.

IV. LOW ENERGY COMPTON SCATTERING

The general picture thus far developed can be used to give valuable insight into many interesting phenomena. We shall examine briefly several simple processes involving the interaction of electrons and radiation. In each case, the first non-vanishing approximation to the energy, or the second non-vanishing approximation to the transition probability will diverge if calculated on the usual theory. The preceding arguments concerning the Lamb shift strongly indicate that the physically real and finite parts of these divergences will always manifest themselves as a spreading out of the electronic charge and current by the fluctuation in position previously calculated.

As the simplest example of a phenomenon in which the position fluctuation modifies a transition probability, we consider the non-relativistic Compton scattering. For simplicity, we consider the effect to be completely classical, since this must give the correct non-relativistic answer. A free electron executes a steady forced oscillation under the action of the incident light wave and emits a scattered light wave of the same frequency. The effect of the position fluctuation is twofold. The electron behaves now like a distributed charge with a mean square radius $\langle(\Delta q)^2\rangle_{\text{Av}}$. It therefore interacts less strongly with the incident wave and radiates a weaker scattered wave. To find the magnitude of this reduction in the interaction between the electron and a light wave, we merely consider the change in the phase factor for the wave introduced by the averaging over the position fluctuation of the electron. This yields

$$\begin{aligned}\langle \exp i\mathbf{k}\cdot(\mathbf{q}+\Delta\mathbf{q})\rangle_{\text{Av}} &= \exp i\mathbf{k}\cdot\mathbf{q}\langle[1+i\mathbf{k}\cdot\Delta\mathbf{q} \\ &\quad -\tfrac{1}{2}(\mathbf{k}\cdot\Delta\mathbf{q})^2+\cdots]\rangle_{\text{Av}} \\ &= \exp i\mathbf{k}\cdot\mathbf{q}[1-\tfrac{1}{6}k^2\langle(\Delta q)^2\rangle_{\text{Av}}], \quad (15)\end{aligned}$$

where $\langle\ \rangle_{\text{Av}}$ indicates an average over the position fluctuation. The correction, as in the case of the Lamb shift, involves the product of the mean square position fluctuation with the Laplacian of the space function describing the interaction. We see that the amplitude of oscillation and the amplitude of the scattered wave will each suffer a fractional reduction equal to

$$\tfrac{1}{6}k^2\langle(\Delta q)^2\rangle_{\text{Av}}. \quad (16)$$

The resultant reduction will be twice as large and the reduction in the scattered intensity or cross section will be twice as large again, so that for the fractional change in cross section, we have

$$\begin{aligned}\frac{\Delta\sigma}{\sigma} &= -\tfrac{2}{3}k^2\langle(\Delta q)^2\rangle_{\text{Av}} \\ &= -\frac{4}{3\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar k}{mc}\right)^2 \log\frac{mc}{\hbar k_0}. \quad (17)\end{aligned}$$

We argue that the angular distribution will remain unchanged, since the scattering retains its dipole character.

There remains the problem of determining the lower cut-off k_0. We observe that frequencies of fluctuation higher than the frequency of the light wave will effectively spread out the scattering charge, while frequencies below this limit will only displace the scattering charge bodily in a random fashion. We therefore place $k_0=k$ and obtain

$$\frac{\Delta\sigma}{\sigma} = -\frac{4}{3\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar k}{mc}\right)^2 \log\frac{mc}{\hbar k}. \quad (18)$$

We see that the correction goes to zero strongly at low frequencies, and that for quanta of two-hundred kilovolts energy, where the formula (18) is probably still adequate, the fractional decrease becomes about one part in two thousand. The increase with energy indicated in (18) cannot be expected to continue much farther, so that an experimental verification of the result (18) seems out of the question.

V. THE INTERACTION BETWEEN A SPIN AND A MAGNETIC FIELD

A further interesting illustration of the utility of the viewpoint of this paper, as well as a useful example of its limitations, can be found in the effect on the Zeeman splitting of an atomic level produced by the interaction of the electron with the radiation field. In this case, fluctuations of the radiation field change the effective potential energy of interaction between the external magnetic field and the spin and orbital magnetic moments of the electron. A simple calculation of the effect can be given by a method analogous to the one previously used. Consider a simple system which consists of an angular momentum with an associated magnetic moment. For definiteness we take the system to be an electron spin which is free except for the action of the fluctuations of the radiation field. Let $(\hbar/2)\boldsymbol{\sigma}$ be the angular momentum operator for the spin. Then the equation of motion for $\boldsymbol{\sigma}$ is

$$\frac{\partial\boldsymbol{\sigma}}{\partial t} = \frac{e}{mc}\mathbf{B}\times\boldsymbol{\sigma}, \quad (19)$$

where $\mathbf{B}$ is the instantaneous magnetic field intensity arising from the field fluctuations. If we consider thee quation of motion for the quantum-mechanical expectation of $\boldsymbol{\sigma}$, we obtain the same equation, but with $\boldsymbol{\sigma}$ interpreted as a unit classical vector.

We solve Eq. (19) by making use of the fact that for vanishing e, $\boldsymbol{\sigma}$ is fixed in direction. We call the fluctuating correction to this fixed vector $\Delta\boldsymbol{\sigma}$ and obtain

$$\Delta\boldsymbol{\sigma}=\frac{e}{mc}\mathbf{B}\times\boldsymbol{\sigma}, \tag{20}$$

where $\boldsymbol{\sigma}$ is constant in time. We follow the same procedure used for obtaining $\langle(\Delta q)^2\rangle_{\text{Av}}$ in the first section of this paper. This yields

$$\langle(\Delta\sigma)^2\rangle_{\text{Av}}=\sigma^2\frac{e^2}{\hbar c}\frac{2}{\pi}\left(\frac{\hbar}{mc}\right)^2\int_0^\kappa k\,dk, \tag{21}$$

where $\langle(\Delta\sigma)^2\rangle_{\text{Av}}$ is the mean square fluctuation in the unit spin vector. The upper limit of the integral has been made κ rather than infinity, but the question of evaluating κ will be deferred for the moment.

We define the mean square angle of fluctuation $\langle(\Delta\theta)^2\rangle_{\text{Av}}$

$$\langle(\Delta\theta)^2\rangle_{\text{Av}}=\frac{\langle(\Delta\sigma)^2\rangle_{\text{Av}}}{\sigma^2}=\frac{2}{\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\frac{\kappa^2}{2}. \tag{22}$$

Now, as in the case of the Lamb shift, we find the effect of this fluctuation on the potential energy of the spin in a magnetic field. This energy is, if the external magnetic field has magnitude B and is in the Z direction,

$$\frac{e\hbar}{2mc}\sigma_z B=\frac{e\hbar B}{2mc}|\boldsymbol{\sigma}|\cos\theta, \tag{23}$$

where θ is the angle between the spin direction and the z axis. We see that the fluctuations merely affect the average value of $\cos\theta$. This effect can easily be found by the use of some simple spherical trigonometry. Let $\bar{\theta}$ be the angle between the z axis and the average spin direction. Let θ be the angle between the instantaneous spin vector and the z axis. Let $\Delta\theta$ and ϕ be the colatitude and azimuth angles of the instantaneous spin vector with respect to the average spin vector. The effect of the fluctuations will then be to replace $\cos\theta$ in (23) by $\langle\cos\theta\rangle_{\text{Av}}$, where the brackets indicate an average over the fluctuations of the spin vector. We have

$$\langle\cos\theta\rangle_{\text{Av}}=\langle\cos\bar{\theta}\cos\Delta\theta+\sin\bar{\theta}\sin\Delta\theta\cos\phi\rangle_{\text{Av}}. \tag{24}$$

The indicated average is easily performed by remembering that the average of $\cos\phi$ must vanish because of the isotropy of the fluctuation. Furthermore, $\Delta\theta$ will be assumed small so that $\cos\Delta\theta$ is well represented by the first two terms in its series expansion. We obtain

$$\begin{aligned}\langle\cos\theta\rangle_{\text{Av}}&=\cos\bar{\theta}[1-\tfrac{1}{2}\langle(\Delta\theta)^2\rangle_{\text{Av}}]\\&=\cos\bar{\theta}\left[1-\frac{e^2}{2\pi\hbar c}\left(\frac{\hbar\kappa}{mc}\right)^2\right].\end{aligned} \tag{25}$$

We then see that the correction to the orientation energy of the spin consists in a reduction in the magnitude of the energy proportional to the energy itself. It is therefore convenient to think of the effect of the interaction with the radiation field as consisting of an alteration of the magnetic moment, μ, associated with the spin. We have

$$\frac{\Delta\mu}{\mu}=-\frac{e^2}{2\pi\hbar c}\left(\frac{\hbar\kappa}{mc}\right)^2. \tag{26}$$

In this case the lower cut-off in the integral over quanta is unimportant, since no contribution arises from small values of k. The magnitude of the effect is, however, correspondingly more sensitive to the value of the upper cut-off. We first consider the case where the magnetic moment μ is associated with the orbital angular momentum of an electron in an atom. Here the upper cut-off is supplied in a natural fashion by the fact that Eq. (19) is valid only for magnetic fields with wave-lengths so long that there is no serious retardation over a distance of the order of the atomic radius. We therefore set

$$\kappa=1/a=(e^2/\hbar c)(mc/\hbar), \tag{27}$$

where a is an average orbital radius for a valence electron, taken to be equal to the Bohr radius. Equation (26) then becomes

$$(\Delta\mu/\mu)=-(1/2\pi)(e^2/\hbar c)^3, \tag{28}$$

so that for an orbital moment, the effect under consideration is completely negligible.

The situation is different if we consider the case of the intrinsic magnetic moment of the electron. Here it must be assumed that the moment can be concentrated to a radius equal roughly to the Compton wave-length of the electron, since a wave packet consisting of positive energy states only and possessing a definite spin can be made to occupy a volume of this radius. Under these conditions, quanta with wave num-

bers higher than $mc/\hbar$ will not act coherently on the spin and we have κ equal to the reciprocal Compton wave-length. Hence, for a spin, Eq. (26) becomes

$$\Delta\mu/\mu = -e^2/2\pi\hbar c, \tag{29}$$

where the exact numerical factor is, of course, not to be taken too seriously. Unfortunately, although the magnitude of the correction seems, from recent molecular beam experiments,[4] to be nearly correct, the sign of the effect should be reversed. This unsatisfactory result of our qualitative considerations is probably to be ascribed to our complete neglect of the relativistic nature of the electron spin. By a careful relativistic calculation, Schwinger[5] has recently shown that the correct sign will indeed follow from a reasonable theory. The effective reduction in the moment which is indicated by our simple non-relativistic theory seems to be more than compensated by an effective increase in the moment which arises, according to Schwinger, from the magnetic interaction of the electron with the filled negative energy states of the vacuum.

VI. QUANTUM-MECHANICAL TREATMENT OF $\langle(\Delta q)^2\rangle_{Av}$ AND JUSTIFICATION OF THE LOWER CUT-OFF

We shall now attempt to give a more formal justification for some of the assertions thus far made, and in particular we propose to give a rigorous quantum-mechanical meaning to the quantity $\langle(\Delta q)^2\rangle_{Av}$, which has thus far been rather loosely treated in an almost purely classical fashion.

We begin by writing the Hamiltonian for an electron moving non-relativistically under the influence of an electrostatic field and the radiation field. We neglect retardation in the interaction between the electron and the radiation field, and assume unit quantization volume for the field

$$H=\frac{p^2}{2m}+V(q)+\sum_k a_k^+ a_k hck - \frac{e}{mc}\sum_k\left(\frac{2\pi\hbar c}{k}\right)^{\frac{1}{2}}(e_k\cdot p)(a_k+a_k^+). \tag{30}$$

Here p and q are the vector momentum and coordinate of the electron, $V(q)$ is the electrostatic potential energy, a_k^+ and a_k are the usual occupation operators for a normal mode of the field with wave number k, and e_k is a unit vector in the direction of polarization of the quantum. The term in the square of the vector potential has been omitted since it does not affect the Lamb shift.

The Hamiltonian (30) has been carefully studied by Pauli and Fierz[6] in connection with the resolution of the infra-red paradox, and we follow their treatment, in its essentials. We first rewrite (30) by completing the square indicated by the last two terms

$$H=\frac{p^2}{2m}+V(\mathbf{q})+\sum_k\left[a_k^+-\frac{e}{mc}\left(\frac{2\pi}{\hbar ck^3}\right)^{\frac{1}{2}}\mathbf{e}_k\cdot\mathbf{p}\right]\times\left[a_k-\frac{e}{mc}\left(\frac{2\pi}{\hbar ck^3}\right)^{\frac{1}{2}}\mathbf{e}_k\cdot\mathbf{p}\right]\hbar ck - \frac{e^2}{m^2c^2}\sum_k\frac{2\pi}{k^2}(\mathbf{e}_k\cdot\mathbf{p})^2. \tag{31}$$

The last term, which has been added to complete the square, is just Bethe's subtraction term, as can be seen by replacing the sum by an integral

$$\frac{e^2}{m^2c^2}\sum_k\frac{2\pi}{k^2}(\mathbf{e}_k\cdot\mathbf{p})^2=\frac{2e^2}{3\pi}\frac{p^2}{m^2c^2}\int_0^\infty dk. \tag{32}$$

We now adopt the point of view that the Hamiltonian (30) is incorrect and should be modified by the addition of the explicitly divergent term (32). The resulting theory should then be free from all divergences of classical origin. We write the Hamiltonian as

$$H=\frac{p^2}{2m}+V(\mathbf{q})+\sum_k A_k^+ A_k hck, \tag{33}$$

with A_k and A_k^+ defined by

$$A_k=a_k-\frac{e}{mc}\left(\frac{2\pi}{\hbar ck^3}\right)^{\frac{1}{2}}\mathbf{e}_k\cdot\mathbf{p},\qquad A_k^+=a_k^+-\frac{e}{mc}\left(\frac{2\pi}{\hbar ck^3}\right)^{\frac{1}{2}}\mathbf{e}_k\cdot\mathbf{p}. \tag{34}$$

[4] J. E. Nafe, E. B. Nelson, and I. I. Rabi, Phys. Rev. **71**, 914 (1947); D. E. Nagle, R. S. Julian, and J. R. Zacharias, Phys. Rev. **72**, 971 (1947); P. Kusch and H. M. Foley, Phys. Rev. **72**, 1256 (1947).

[5] J. Schwinger, Phys. Rev. **73**, 416 (1948).

[6] W. Pauli and M. Fierz, Nuovo Cimento **15**, 167 (1938).

If we define new variables P and Q by

$$\mathbf{P}=\mathbf{p},$$
$$\mathbf{Q}=\mathbf{q}+\frac{ie}{mc}\sum_k\left(\frac{2\pi\hbar}{ck^3}\right)^{\frac{1}{2}}\mathbf{e}_k(a_k-a_k^+), \tag{35}$$

we see that the capitalized variables are related to the small variables by a contact transformation. We write the Hamiltonian in terms of the new variables, replacing capital letters by small letters

$$H=\frac{p^2}{2m}+V\left(\mathbf{q}-\frac{ie}{mc}\sum_k\left(\frac{2\pi\hbar}{ck^3}\right)^{\frac{1}{2}}\mathbf{e}_k(a_k-a_k^+)\right)+\sum_k a_k^+a_k\hbar ck. \tag{36}$$

It is finally convenient to change to field variables which are the real coordinates and momenta of harmonic oscillators. We thus obtain

$$H=\frac{p^2}{2m}+V(\mathbf{q}+\Delta\mathbf{q})+\sum_k\tfrac{1}{2}(P_k^2+c^2k^2Q_k^2), \tag{37}$$[7]

with

$$\Delta\mathbf{q}=\frac{e}{mc}(4\pi)^{\frac{1}{2}}\sum_k\mathbf{e}_k\frac{Q_k}{k}. \tag{38}$$

The general character of the consequences of this Hamiltonian is easy to deduce. We assume that for all quanta with frequencies above k_0, the corresponding radiation oscillators execute their unperturbed motion. Quanta below this cut-off will be arbitrarily excluded for the time being, and it will be further assumed, in order to avoid the necessity of discussing the infra-red paradox, that the limit k_0 is placed sufficiently high that no quanta can be emitted. The effect of the second term in (37) will then be only to produce elastic deflections of the electron. We see that in a calculation of the probability of such a deflection, by the use of the Born approximation, the potential energy term in (37) is effectively replaced by its average over the coordinates of all the radiation oscillators. This average can be easily calculated by considering the Fourier transform of $V(\mathbf{q})$. We take

$$V(\mathbf{q})=\int d\boldsymbol{\lambda}U(\boldsymbol{\lambda})\exp i\boldsymbol{\lambda}\cdot\mathbf{q},$$
$$V(\mathbf{q}+\Delta\mathbf{q})=\int d\boldsymbol{\lambda}U(\boldsymbol{\lambda})\exp i\boldsymbol{\lambda}\cdot\mathbf{q}\times\prod_k\exp\left(\frac{ie}{mc}(4\pi)^{\frac{1}{2}}(\mathbf{e}_k\cdot\boldsymbol{\lambda})\frac{Q_k}{k}\right). \tag{39}$$

To obtain the effective potential energy, we must average the product at the end of the integrand over the unperturbed Gaussian wave functions for the ground states of the radiation oscillators. The integrals are easily done, yielding

$$\begin{aligned}\langle V(\mathbf{q}+\Delta\mathbf{q})\rangle_{\text{Av}}&=\int d\boldsymbol{\lambda}U(\boldsymbol{\lambda})\exp i\boldsymbol{\lambda}\cdot\mathbf{q}\times\prod_k\exp\left(-\pi(\mathbf{e}_k\cdot\boldsymbol{\lambda})^2\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\frac{1}{k^3}\right)\\&=\int d\boldsymbol{\lambda}U(\boldsymbol{\lambda})\exp i\boldsymbol{\lambda}\cdot\mathbf{q}\times\exp\left(-\frac{1}{3\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\times\int_{k_0}^{\kappa}\frac{dk}{k}\lambda^2\right).\end{aligned} \tag{40}$$

This expression can now be rewritten in several interesting ways

$$\begin{aligned}\langle V(\mathbf{q}+\Delta\mathbf{q})\rangle_{\text{Av}}&=\int d\boldsymbol{\lambda}U(\boldsymbol{\lambda})\exp i\boldsymbol{\lambda}\cdot\mathbf{q}\times\exp-\tfrac{1}{6}\lambda^2\langle(\Delta q)^2\rangle_{\text{Av}}\\&=\exp(\tfrac{1}{6}\langle(\Delta q)^2\rangle_{\text{Av}}\nabla^2)\,V(\mathbf{q}).\end{aligned} \tag{41}$$

The differential operator appearing in (41) is in reality an integral operator which can easily be written explicitly as

$$\langle V(\mathbf{q}+\Delta\mathbf{q})\rangle_{\text{Av}}=\left[\frac{3}{2\pi\langle(\Delta q)^2\rangle_{\text{Av}}}\right]^{\frac{3}{2}}\int d\mathbf{q}'\,V(\mathbf{q}')\times\exp\left(-\frac{3|\mathbf{q}-\mathbf{q}'|^2}{2\langle(\Delta q)^2\rangle_{\text{Av}}}\right). \tag{42}$$

The displacement kernel occurring in (42) is clearly just the distribution function for the quantity $\Delta\mathbf{q}$, previously assumed in Eq. (11). It is interesting to consider from our present point

[7] A similar transformation of the Hamiltonian seems to have been used by H. A. Kramers (unpublished).

of view the somewhat unexpected results encountered by Pauli and Fierz in their treatment of the infra-red paradox by means of the Hamiltonian (37). They found that the correction of order e^2 to the elastic scattering of a particle by the potential V diverged if no upper cut-off were made in the sum over quanta. They were able to show, however, that if the correction were calculated exactly from the approximate Hamiltonian, the elastic scattering cross section became identically zero, even though no quanta could actually be emitted. Both results were difficult to understand at the time because the removal of the electromagnetic self-energy had presumably rendered the theory convergent. We now see that the residual divergence is just that occurring at the upper limit of the integral over quanta in our expression for the mean square fluctuation in position of the electron. To the order e^2, the resulting infinite position fluctuation will give rise to an infinite correction to the scattering potential and therefore to the cross section. On the other hand, the integral operator (42) really has the effect of reducing the potential to zero if $\langle(\Delta q)^2\rangle_{Av}$ is taken to be infinite. This gives zero scattering cross section in an exact calculation. Both difficulties will clearly be removed, as indicated in the last section of this paper, by the inclusion of recoil, which gives an automatic upper cut-off. It is, of course, necessary, as has been pointed out by Lewis,[8] to make the appropriate subtraction for the electromagnetic self-energy just as has been assumed in obtaining the Hamiltonian (37).

We now turn to the consideration of the significance of the lower cut-off k_0. From the Hamiltonian (37), we can immediately arrive at a formal justification for our earlier intuitive arguments. We consider, for simplicity, the ground state of the system, which for vanishing e goes over into a state whose wave function is the product of the ground state wave function of the atom with the ground state wave functions for all the radiation oscillators. In order to obtain the Lamb shift for this state, we must calculate the energy change to the order e^2 produced by the term added to $\mathbf{q}$ in the argument of the potential energy. This can easily be done and leads in fact just to Bethe's result for the energy shift, as it must. For our purposes, however, we observe that the radiation oscillators can be divided into those with frequencies higher than the characteristic frequency of the electronic motion and those with lower frequencies. The high frequency oscillators are not particularly affected by the coupling with the electron, and the part of the energy shift arising from them is to be considered as that due to averaging the potential energy over the unperturbed ground state wave functions of these oscillators. The low frequency oscillators are, according to the original form of the Hamiltonian, to be thought of as driven by the electronic motion rather than the reverse.

To demonstrate the correctness of this viewpoint, we make a Taylor expansion of the potential energy to the order e^2 and write the terms which appear as perturbations

$$V(\mathbf{q}+\Delta\mathbf{q})-V(\mathbf{q}) = \frac{e}{mc}(4\pi)^{\frac{1}{2}}\sum_k \frac{1}{k}Q_k(\mathbf{e}_k\cdot\nabla)V(\mathbf{q}) + \frac{1}{2}\left(\frac{e}{mc}\right)^2 4\pi\Big[\sum_k Q_k(\mathbf{e}_k\cdot\nabla)\Big]^2 V(\mathbf{q}). \quad (43)$$

The energy correction to the order e^2 can be obtained by considering the effect of the first term on the right-hand side of (43) in the second order of the perturbation theory and the second term in the first order. We obtain

$$\Delta E = 4\pi\left(\frac{e}{mc}\right)^2\sum_k\sum_n \frac{\hbar}{2ck^3}\frac{|(0|e_k\cdot\nabla V|n)|^2}{E_0-E_n-\hbar ck} + \tfrac{1}{2}4\pi\left(\frac{e}{mc}\right)^2\sum_k\frac{\hbar}{2ck^3}(0|(e_k\cdot\nabla)^2V|0), \quad (44)$$

where the index n refers to the nth atomic state, and the unperturbed atomic state is denoted by the index 0. We then replace the sums over radiation oscillators by integrals and do the integrals over the angles of k and the sum over polarization directions. The resulting expression is

$$\Delta E = \frac{1}{3\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\int_0^\kappa\frac{dk}{k}\left\{(0|\nabla^2V|0) + 2\sum_n\frac{(0|\nabla V|n)\cdot(n|\nabla V|0)}{E_0-E_n-\hbar ck}\right\}. \quad (45)$$

The first term in (45) is immediately recognizable as the energy shift which would be ob-

[8] H. W. Lewis, Phys. Rev. **73**, 173 (1948).

tained by our original semiclassical argument if the position fluctuation were taken to be that for a free electron. The assumption of course is always to be made that the formally infinite upper limit of integration is in reality the reciprocal Compton wave-length. The second term can be given a simple interpretation by studying its behavior for small and large values of k. We first assume that $hck \ll |E_0 - E_1|$, where E_1 is the energy of the lowest intermediate atomic state that occurs in the sum over n. We can then expand the integrand of the second term in powers of k

$$2\sum_n \frac{(0|\nabla V|n)\cdot(n|\nabla V|0)}{E_0 - E_n - hck} = 2\sum_n (0|\nabla V|n)\cdot(n|\nabla V|0)\left\{\frac{1}{E_0 - E_n} + \frac{2\hbar ck}{(E_0 - E_n)^2} + 0(k^2)\right\}. \quad (46)$$

The first two terms in (46) can be simplified by the use of the equations of motion and the commutation relations. We obtain

$$2\sum_n \frac{(0|\nabla V|n)\cdot(n|\nabla V|0)}{E_0 - E_n - hck} = -(0|\nabla^2 V|0) + \frac{2ck}{\hbar}(0|p^2|0) + 0(k^2). \quad (47)$$

We then see that the apparent logarithmic divergence at the lower limit of the integral in (45) is removed. The largest surviving contribution to the integrand for low frequency quanta is just the low frequency contribution to the classical electromagnetic self-energy. This cancellation will approximately persist until the series expansion used in Eq. (46) becomes poorly convergent, that is until the quantum frequency becomes comparable with the orbital frequency of the electron. For frequencies higher than this the situation will become quite complicated until the quantum energy becomes larger than the highest energy effectively occurring in the sum over intermediate atomic states. For quantum energies so high that the sum in the second term of (45) effectively converges before the denominator begins to differ much from hck, we see that the importance of this term in comparison with the first term decreases with increasing k. As a result of this limiting behavior of the second term for small k and large k, it is convenient to represent the energy shift by the expression

$$\Delta E = \frac{1}{3\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2 \int_{k_0}^{\kappa} \frac{dk}{k}(0|\nabla^2 V|0), \quad (48)$$

where the lower limit k_0 is adjusted to give the correct value for (45).

This effective lower cut-off can apparently be evaluated only by doing the indicated sum, but a useful general property can easily be deduced. For a harmonic oscillator, the sum over states reduces to a single term. The atomic energy difference occurring in this term is just that corresponding to the orbital frequency of the electron; the transition between the two limiting types of behavior for the sum takes place for a quantum frequency equal to the orbital frequency, and ck_0 will therefore be very close to the orbital frequency.

This simple behavior holds only for a harmonic oscillator. As the potential becomes more concentrated near the center (approaching the Coulomb potential, for example), the matrix elements of ∇V appearing in the sum over states will become important for states lying higher and higher above the unperturbed atomic state. The first terms in the sum will always have energy differences corresponding to the orbital frequency of the electron, but the first terms become relatively less and less important as we approach the Coulomb potential. The quantity k_0, which marks the transition between the two limiting forms for the sum, will accordingly rise higher and higher above the orbital frequency. A careful evaluation of the effective lower cut-off for the case of the $2S$ level of hydrogen gives, according to Bethe,[1] a value of 17.8 Rydbergs or approximately twenty times the orbital frequency. This seemingly high value is now not surprising in view of the foregoing arguments.

VII. JUSTIFICATION OF THE UPPER CUT-OFF

In this section we shall attempt to give a quantum-mechanical derivation of the expression for the mean square fluctuation in position. We shall include the effect of retardation but not the effect of relativity. In this way, the existence of the

upper cut-off in the integral over quantum energies will be given some formal support. We assume the complete non-relativistic Hamiltonian for a spinless charged particle coupled to the radiation field, but otherwise free

$$H=\frac{p^2}{2m}+\sum_k a_k^+ a_k \hbar ck + \frac{e}{mc}\sum_k \left(\frac{2\pi\hbar c}{kV}\right)^{\frac{1}{2}} (\mathbf{e}_k\cdot\mathbf{p}) \times (a_k \exp i\mathbf{k}\cdot\mathbf{q}+a_k^+ \exp -i\mathbf{k}\cdot\mathbf{q}) + \frac{e^2}{2mc^2}\left[\sum_k \left(\frac{2\pi\hbar c}{kV}\right)^{\frac{1}{2}} \mathbf{e}_k \times (a_k \exp i\mathbf{k}\cdot\mathbf{q}+a_k^+ \exp -i\mathbf{k}\cdot\mathbf{q})\right]^2. \quad (49)$$

In analogy with the semiclassical derivation given in the first part of this paper, we assume an electron at rest and calculate its motion in the first approximation under the action of the coupling with the field. The unperturbed wave functions for the problem are

$$\frac{\exp(i\mathbf{P}\cdot\mathbf{q})}{(V)^{\frac{1}{2}}}(\mathbf{k}_1, \mathbf{k}_2, \cdots), \quad (50)$$

where $\mathbf{P}$ is the wave vector for the wave function of a free electron with definite momentum and $(k_1, k_2, \cdots)$ is the wave function for the radiation field with the indicated quanta present. The zero-order wave function has no quanta present and has in it a range of electron momenta, so that the position of the electron can be approximately given. We have

$$\psi=\sum_P A_P \frac{\exp(i\mathbf{P}\cdot\mathbf{q})}{(V)^{\frac{1}{2}}}(\cdots). \quad (51)$$

The electron density, averaging over the coordinates of the radiation oscillators, is

$$\langle|\psi|^2\rangle_{Av}=\sum_P\sum_{P'} A_{P'}^* A_P \frac{\exp(i(\mathbf{P}-\mathbf{P}')\cdot\mathbf{q})}{V}. \quad (52)$$

We now assume that the coupling between the electron and the radiation field is turned on. Each wave function occurring in the sum (51) will be perturbed, and the electron density (52) will be accordingly altered. We proceed to calculate the perturbed density to the order e^2. The third term in our Hamiltonian will enter to the first order in the correction to the wave function. It cannot enter to the second order, since this would either lead back to the zero-order wave function, or to one referring to two quanta, which is orthogonal to the zero-order function and will therefore give no contribution in the averaging over the radiation field coordinates. The last term in the Hamiltonian gives no contribution for the same reason. It is interesting to note that any subtraction term added to the Hamiltonian to render the theory convergent will give no contribution for the same reason again. A second-order correction to the wave function does, however, arise from the renormalization necessitated by the first-order correction.

The wave function referring to definite electron momentum becomes

$$\frac{\exp i\mathbf{P}\cdot\mathbf{q}}{(V)^{\frac{1}{2}}}(\cdots)+\frac{e\hbar}{mc}\sum_k \left(\frac{2\pi\hbar c}{kV}\right)^{\frac{1}{2}} \frac{(\mathbf{e}_k\cdot\mathbf{P})}{E_P-E_{P-k}-\hbar ck} \times \frac{\exp(i(\mathbf{P}-\mathbf{k})\cdot\mathbf{q})}{(V)^{\frac{1}{2}}}(\mathbf{k})-\frac{e^2\hbar^2}{2m^2c^2}\sum_k \frac{2\pi\hbar c}{kV} \times \frac{(\mathbf{e}_k\cdot\mathbf{P})^2}{(E_P-E_{P-k}-\hbar ck)^2}\frac{\exp(i\mathbf{P}\cdot\mathbf{q})}{(V)^{\frac{1}{2}}}(\cdots). \quad (53)$$

The averaged electron density then becomes

$$\langle|\psi|^2\rangle_{Av}=\sum_P\sum_{P'} A_{P'}^* A_P \frac{\exp(i(\mathbf{P}-\mathbf{P}')\cdot\mathbf{q})}{V} \times\left\{1-\frac{e^2\hbar^2}{2m^2c^2}\sum_k \frac{2\pi\hbar c}{kV}\left(\frac{\mathbf{e}_k\cdot\mathbf{P}}{E_P-E_{P-k}-\hbar ck}-\frac{\mathbf{e}_k\cdot\mathbf{P}'}{E_{P'}-E_{P'-k}-\hbar ck}\right)^2\right\}. \quad (54)$$

We now assume that P and P' are always small compared with the reciprocal Compton wavelength. The energy denominators become

$$E_P-E_{P-k}-\hbar ck=\frac{\hbar^2P^2}{2m}-\frac{\hbar^2(\mathbf{P}-\mathbf{k})^2}{2m}-\hbar ck = -\hbar ck\left(1+\frac{\hbar k}{2mc}-\frac{\hbar\mathbf{k}\cdot\mathbf{P}}{mck}\right). \quad (55)$$

The last term in (55) is always very small compared with the first term, so we shall ignore it. The electron density becomes, changing the sum

to an integral and doing the sum over polarizations and the integration over angles,

$$\langle|\psi|^2\rangle_{\text{Av}}=\sum_P\sum_{P'}A_{P'}{}^*A_P\frac{\exp(i(\mathbf{P}-\mathbf{P}')\cdot\mathbf{q})}{V}\times\left\{1-\frac{1}{3\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2(\mathbf{P}-\mathbf{P}')^2\times\int_0^\infty\frac{dk}{k\left(1+\frac{\hbar k}{2mc}\right)^2}\right\}. \quad (56)$$

This can be rewritten in a more transparent form

$$\langle|\psi|^2\rangle_{\text{Av}}=\left\{1+\frac{2}{\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\times\int_0^\infty\frac{dk}{k\left(1+\frac{\hbar k}{2mc}\right)^2}\,e^{\frac{1}{6}\nabla^2}\right\}\times\sum_P\sum_{P'}A_{P'}{}^*A_P\frac{\exp(i(\mathbf{P}-\mathbf{P}')\cdot\mathbf{q})}{V}. \quad (57)$$

From this we see that the effect of the coupling between electron and radiation field is to spread out the electron density function, the mean square spreading distance being

$$\langle(\Delta q)^2\rangle_{\text{Av}}=\frac{2}{\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\int_0^\infty\frac{dk}{k\left(1+\frac{k}{2\kappa}\right)^2}=\frac{2}{\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\left\{\int_{k_0}^\infty\frac{dk}{k\left(1+\frac{k}{2\kappa}\right)^2}+\int_0^{k_0}\frac{dk}{k\left(1+\frac{k}{2\kappa}\right)^2}\right\}. \quad (58)$$

We make our usual assumption that only frequencies above some lower cut-off k_0 are effective when the electron is bound, and therefore omit the second term in (58). The integral over k can be performed without difficulty, yielding

$$\langle(\Delta q)^2\rangle_{\text{Av}}=\frac{2}{\pi}\frac{e^2}{\hbar c}\left(\frac{\hbar}{mc}\right)^2\log\frac{2\kappa}{\epsilon k_0}, \quad (59)$$

where ϵ is the base of the natural logarithms. We therefore see that the inclusion of the retardation leads to an upper cut-off which is about three quarters of the crude value (κ) originally assumed.[9]

The derivation just given has some attractive features. It gives a convergent result for the physically meaningful part of the reaction of the field on the electron, without the necessity of subtracting two infinite terms. The result (59) suffers, however, from the obvious disadvantage that a non-relativistic Hamiltonian was used to obtain convergence in a region where the non-relativistic assumption is clearly seriously in error. It would seem to be of the greatest interest to apply the method just used to the calculation of $\langle(\Delta q)^2\rangle_{\text{Av}}$ in the hole theory. Such a procedure is seen to be scarcely unambiguous, although it is apparently again possible to obtain the physically interesting results without the introduction of the usual subtraction terms. It is hoped that this point can be treated in detail in a later paper.

VIII. ACKNOWLEDGMENT

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[9] W. E. Lamb has pointed out (unpublished) that the inclusion of recoil in a non-relativistic calculation of the line shift in hydrogen gives a convergent result.