ch8 The Gamma Function

8.1 Definitions, Simple Properties

- Infinite Limit (Euler)

The first definition is:

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z \qquad z \neq 0, -1, -2, -3, \ldots$$

Replacing z with $z+1$,

$$\Gamma(z+1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{(z+1)(z+2)\cdots(z+n+1)} n^{z+1}$$

$$= \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots\cdot n}{z(z+1)(z+\cdots)} n^z \cdot \frac{n\cdot z}{(z+n+1)}$$

$$= z\Gamma(z) \qquad \therefore \quad \Gamma(z+1) = z\Gamma(z)$$

Now,

$$\Gamma(1) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{1\cdot 2\cdot 3\cdots n(n+1)} n^1 = 1$$

and

$$\Gamma(2) = \Gamma(1+1) = 1\Gamma(1) = 1$$

$$\Gamma(3) = \Gamma(2+1) = 2\Gamma(2) = 2\cdot\Gamma(1) = 1$$

$$\cdots$$

$$\Gamma(n) = (n-1)(n-2)\cdots 2\cdot 1 = (n-1)!$$

- Definite integral (Euler integral)

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1}\, dt, \qquad \mathrm{Re}\, z > 0$$

If we let $t = x^2$, $dt = 2x\,dx$

$$\Gamma(z) = \int_0^\infty e^{-x^2}(x^2)^{z-1}(2x)\,dx = 2\int_0^\infty e^{-x^2} x^{2z-1}\,dx = 2\int_0^\infty e^{-t^2} t^{2z-1}\,dt$$

or, letting $e^{-t} = x$, $-t = \ln x$, $-e^{-t}dt = dx$

$$\Gamma(z) = \int_1^0 e^{-t}\left[\ln\left(\frac{1}{x}\right)\right]^{z-1}(-1)e^{+t}\,dx$$

$$= \int_0^1 \left[\ln\left(\frac{1}{x}\right)\right]^{z-1} dx = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt$$

From $\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1}\, dt$,

if $z = \frac{1}{2}$, $\Gamma(\frac{1}{2}) = 2\int_0^\infty e^{-t^2} dt = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$ (Gauss error integral)

· two definitions are equal to each other:

Consider

$$F(z,n) = \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\, dt, \quad Re(z) > 0$$

n: positive integer

Since $\lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n \equiv e^{-t}$ from the definition of the exponential

$$\lim_{n\to\infty} F(z,n) = F(z,\infty) = \int_0^\infty e^{-t} t^{z-1}\, dt \equiv \Gamma(z)$$

Returning to $F(z,n)$: let $u = t/n$

$$F(z,n) = \int_0^1 (1-u)^n u^{z-1}\, du$$

Integrating by parts, we obtain

$$\frac{F(z,n)}{n^z} = \underbrace{(1-u)^n \frac{u^z}{z}\Big|_0^1}_{=0} - \int_0^1 (-1)\, n\, (1-u)^{n-1} \frac{u^z}{z}\, du$$

$$= \frac{n}{z}\int_0^1 (1-u)^{n-1} u^z\, du$$

Repeating this,

$$F(z,n) = n^z \frac{n(n-1)\cdots 1}{z(z+1)\cdots(z+n-1)} \int_0^1 u^{z+n-1}\, du$$

$$= \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)\cdots(z+n)}\, n^z$$

identical to the (Euler's definition)

Therefore, $\lim_{n\to\infty} F(z,n) = F(z,\infty) \equiv \Gamma(z)$

· Infinite Product (Weierstrass)

The 3rd definition (Weierstrass' form) is

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$$

where γ is Euler-Mascheroni constant , $\gamma = 0.5772156619\cdots$

This form can be derived from the original definition

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdot 4 \cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z$$

dividing each term in the denominator by one in the numerator

$$= \lim_{n\to\infty} \frac{1}{z(z+1)(\frac{z}{2}+1)\cdots(\frac{z}{n}+1)} n^z$$

$$= \lim_{n\to\infty} \frac{1}{z} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right)^{-1} n^z$$

using the fact that $n^{-z} = e^{(-\ln n)z}$

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} e^{(-\ln n)z} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right)$$

also, multiply and divide the following eq:

$$e^{[(1+\frac{1}{2}+\frac{1}{3}+\cdots \frac{1}{n})z]} = \prod_{m=1}^{n} e^{z/m}$$

we get (taking $n\to\infty$ as well)

$$\frac{1}{\Gamma(z)} = z \left\{ \lim_{n\to\infty} e^{[(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\ln n)z]} \right\}$$

$$\times \left[\lim_{n\to\infty} \prod_{m=1}^{n} \left(1+\frac{z}{m}\right) e^{-z/m} \right]$$

we learned that $\gamma = \lim_{n\to\infty} \left(\sum \frac{1}{n} - \ln n \right)$,

above equation becomes

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right) e^{-z/n}$$

In Section 5.11 we showed that

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$$

Alternatively, we can start from the product of Euler integrals

$$\Gamma(z+1)\Gamma(1-z) = \int_0^\infty s^z e^{-s} ds \int_0^\infty t^{-z} e^{-t} dt$$

let $u = s+t$, $v = s/t$ $\rightarrow$ $u = t(v+1)$, $t = \frac{u}{v+1}$, $s = \frac{uv}{1+v}$

The integral becomes

$$\int_0^\infty \left(\frac{uv}{1+v}\right)^z e^{-\frac{uv}{1+v}} \frac{u\,dv}{1+v} \int_0^\infty \left(\frac{u}{v+1}\right)^{-z} e^{-\frac{u}{v+1}} \frac{du}{v+1}$$

$$= \int_0^\infty v^z \frac{dv}{(v+1)^2} \int_0^\infty e^{-\frac{u}{1+v} - \frac{uv}{1+v}} u\,du$$

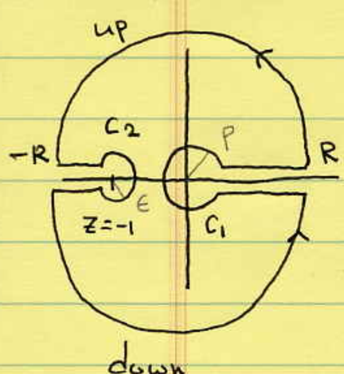
$$-\frac{u}{1+v} - \frac{uv}{1+v} = \frac{-u(1+v)}{1+v} = -u$$

$$= \int_0^\infty v^z \frac{dv}{(v+1)^2} \int_0^\infty e^{-u} u\,du$$

Note: $\int_0^\infty e^{-u} u\,du = (-e^{-u}u)\Big|_0^\infty - \int_0^\infty (-e^{-u})\,du$

$$= \int_0^\infty e^{-u} du = -e^{-u}\Big|_0^\infty = 1$$

1st term: $\int_0^\infty v^z \frac{dv}{(v+1)^2}$



$z=0$: branch point

positive x-axis : cut line

$$\oint \frac{z^a}{(1+z)^2} dz = \oint_{C_1} \frac{(\rho e^{i\theta})^a i\rho e^{i\theta} d\theta}{(\rho e^{i\theta}+1)^2} + \int_\rho^R \frac{x^a dx}{(x+1)^2}$$

$$+ \oint_{up} \frac{(Re^{i\theta})^a iRe^{i\theta} d\theta}{(Re^{i\theta}+1)} + \int_R^{-1-\epsilon} \frac{(Re^{i\pi})^a e^{i\pi} dx}{(Re^{i\pi}+1)^2}$$

$$+ \oint_{C_2} \frac{(\epsilon e^{i\theta}-1)^a i\epsilon e^{i\theta} d\theta}{(1+\epsilon e^{i\theta})^2} + \int_{-1-\epsilon}^{-R} \frac{(Re^{i\pi})^a}{(Re^{i\pi}+1)^2} e^{i\pi} dx$$

$$+ \oint_{down} \frac{(Re^{i\theta})^a iRe^{i\theta} d\theta}{(Re^{i\theta}+1)^2} + \int_R^\rho \frac{(Re^{2\pi i})^a e^{2\pi i} dx}{(Re^{2\pi i}+1)^2}$$

$$= \left(1 - e^{2\pi a i}\right) \int_0^\infty \frac{x^a}{(x+1)^2}\,dx$$

Note $\dfrac{z^a}{(z+1)^2} = \dfrac{z^a}{(z-e^{i\pi})^2}$ and $f^{(n)}(z_0) = \dfrac{n!}{2\pi i}\oint \dfrac{f(z)\,dz}{(z-z_0)^{n+1}}$

$$\therefore \left(1 - e^{2\pi a i}\right)\int_0^\infty \frac{x^a}{(x+1)^a}\,dx = 2\pi i \cdot a\left(e^{\pi i}\right)^{a-1}$$

$$= -2\pi i\, a\, e^{\pi i a}$$

$$\therefore \int_0^\infty \frac{x^a}{(x+1)^a}\,dx = \frac{\pi a}{\sin \pi a}$$

$$\rightarrow \Gamma(z+1)\Gamma(1-z) = \frac{\pi z}{\sin \pi z}$$

One can establish Legendre's duplication formula

$$\Gamma(1+z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\,\Gamma(2z+1)$$

- Factorial Notation

we write $\displaystyle\int_0^\infty e^{-t} t^z\,dt \equiv z!$ $\quad R(z) > -1$

So, $\Gamma(z) = (z-1)!$ or $\Gamma(z+1) = z!$

If $z = n$, a positive integer, we know that $z! = n! = 1 \cdot 2 \cdot 3 \cdots n$

we know that

$\Gamma(z+1) = z\,\Gamma(z)$. so, it becomes

$$(z-1)! = \frac{z!}{z}$$

$z = 1$ $\quad 0! = \dfrac{1!}{1} = 1$

$z = 0$ $\quad (-1)! = \dfrac{0!}{0} = +\infty$

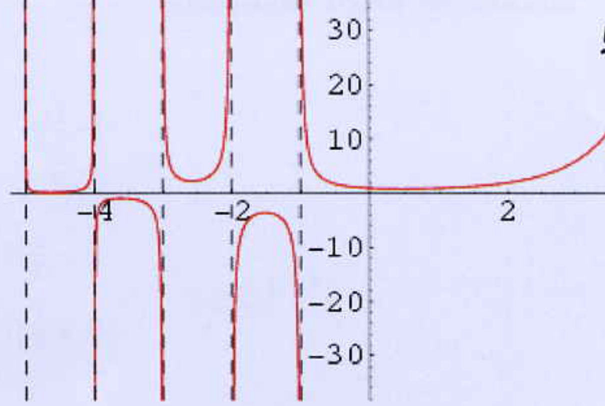
$z = -1$ $\quad (-2)! = \dfrac{(-1)!}{(-1)} = -\infty$

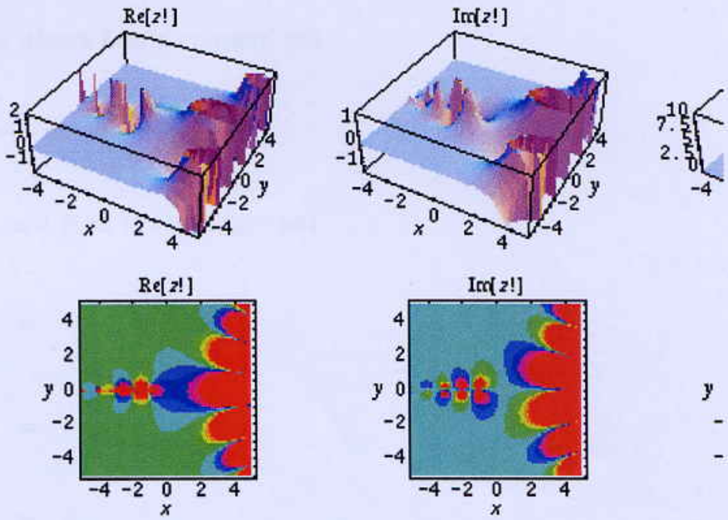
$z = -2$ $\quad (-3)! = \dfrac{(-2)!}{(-2)} = +\infty$

$\vdots$

giving $n! = \pm\infty$ for n, a negative integer

Since $\Gamma(z+1)\Gamma(1-z) = \dfrac{\pi z}{\sin \pi z}$, we get $z!(-z)! = \dfrac{\pi z}{\sin \pi z}$





	Min	Max
Re	-5	5
Im	-5	5

Replot

By noting that

$$n! = \Gamma(n+1),$$

where $\Gamma(n)$ is the gamma function for integers n, the definition can be generali:

$$z! \equiv \Gamma(z+1) \equiv \int_0^\infty e^{-t} t^z \, dt.$$

This defines $z!$ for all complex values of z, except when n is a negative integer, infinity.

Using the identities for gamma functions, the values of $\left(\frac{1}{2}\, n\right)!$ (half integral val

- Double Factorial Notation

we encounter products of the odd positive integers and
" even "

$$1 \cdot 3 \cdot 5 \cdots (2n+1) = (2n+1)!!$$

$$2 \cdot 4 \cdot 6 \cdots 2n = 2n!!$$

now, $(2n)!! = 2^n n!$ and $(2n+1)!! = \dfrac{(2n+1)!}{2^n n!}$

$\ulcorner$proof) $2^n \cdot n! = 2^n \cdot 1 \cdot 2 \cdot 3 \cdots n = 2 \cdot 4 \cdot 6 \cdots 2n = (2n)!!$

$$(2n+1)!! \cdot 2^n n! = 1 \cdot 3 \cdot 5 \cdots (2n+1) \cdot 2^n \cdot 1 \cdot 2 \cdot 3 \cdots n$$

$$= 1 \cdot 3 \cdot 5 \cdots (2n+1) \cdot 2 \cdot 4 \cdot 6 \cdots 2n = (2n+1)! \lrcorner$$

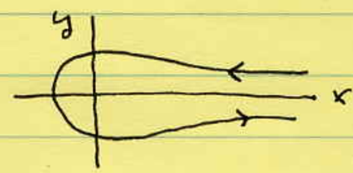
We define $(-1)!! = 1$ (special case)

- Integral Representation

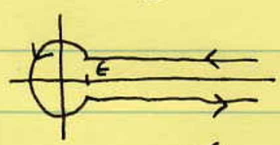
An integral representation that is useful in developing asymptotic series for the Bessel function

$$\int_c e^{-z} z^\nu \, dz = \left(e^{2\pi i \nu} - 1 \right) \Gamma(\nu+1)$$

where c is the contour



Now, deforming the contour as



$z = \epsilon e^{i\theta}$

$$\int_c e^{-z} z^\nu dz = \int_\infty^\epsilon e^{-x} x^\nu dx + \oint_{\substack{\text{small}\\\text{circle}}} e^{-\epsilon e^{i\theta}} (\epsilon e^{i\theta}) (i\epsilon) e^{i\theta} d\theta$$

$$+ \int_\epsilon^\infty e^{-x} x^\nu (e^{2\pi i})^\nu dx$$

$\epsilon \to 0$

$$= \left(e^{2\pi i \nu} - 1\right) \int_0^\infty e^{-x} x^\nu dx = \left(e^{2\pi i \nu} - 1\right) \Gamma(\nu+1)$$

Now, $\int_C e^{-z}(-z)^{\nu}\,dz = ?$

$(-z)^{\nu} = (z \cdot e^{-\pi i})$. Now we can get for the

integral $\leftarrow$, the angle becomes $e^{-i\pi\nu}$

" $\rightarrow$ " $e^{i\pi\nu}$

$$\therefore \int_C e^{-z}(-z)^{\nu}\,dz = (e^{i\pi\nu} - e^{-i\pi\nu})\,\Gamma(\nu+1)$$

$$= 2i\sin(\nu\pi)\,\Gamma(\nu+1)$$

8.2 DiGamma and polygamma functions

· Digamma Functions

From $\Gamma(z+1) = z\Gamma(z) = \lim_{n\to\infty} \frac{n!}{(z+1)(z+2)\cdots(z+n)} n^{z}$

take the natural logarithm of the factorial function

$$\ln\Gamma(z+1) = \lim_{n\to\infty}\left[\ln n! + z\ln n - \ln(z+1) - \ln(z+2) - \cdots - \ln(z+n)\right]$$

Differentiating with respect to z, we obtain

$$\frac{d}{dz}\ln\Gamma(z+1) \equiv \psi(z+1) = \lim_{n\to\infty}\left(\ln n - \frac{1}{z+1} - \frac{1}{z+2} - \cdots - \frac{1}{z+n}\right)$$

This function $\psi(z+1)$ is called digamma function

Since we know that the Euler-Mascheroni constant is

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \ln n\right),$$

the digamma function becomes

$$\psi(z+1) = \lim_{n\to\infty}\left(\ln n - \sum_{m=1}^{n}\frac{1}{m} + \sum_{m=1}^{n}\frac{1}{m} - \sum_{m=1}^{n}\frac{1}{z+m}\right)$$

$$= -\gamma + \lim_{n\to\infty}\sum_{m=1}^{n}\left(\frac{1}{m} - \frac{1}{z+m}\right)$$

$$= -\gamma + \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{z+n}\right) = -\gamma + \lim_{n\to\infty}\frac{z}{n(z+n)}$$

Clearly, $\psi(1) = -\gamma = -0.577\,215\,664\,901\cdots$

· Polygamma Function

The digamma function may be differentiated repeatedly, giving rise to the polygamma function:

$$\psi^{(m)}(z+1) = \frac{d^m}{dz^m}\psi(z+1)$$

$$= \frac{d^m}{dz^m}\left(-\gamma + \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{z+n}\right]\right)$$

$$= \frac{d^m}{dz^m}\sum_{n=1}^{\infty}\left(-\frac{1}{z+n}\right) = \frac{d^{m-1}}{dz^{m-1}}(-1)^2\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}$$

$$= \frac{d^{m-2}}{dz^{m-2}}(-1)^3\sum_{n=1}^{\infty}\frac{2}{(z+n)^3}$$

$$= (-1)^{m+1}\, m!\sum_{n=1}^{\infty}\frac{1}{(z+n)^{m+1}} \quad \text{where } m=1,2,3,\cdots$$

The Riemann zeta function was

$$\zeta(m) \equiv \sum_{n=1}^{\infty}\frac{1}{n^m}, \quad \text{we get}$$

$$\psi^{(m)}(1) = (-1)^{m+1}\, n!\sum_{n=1}^{\infty}\frac{1}{n^{m+1}} = (-1)^{m+1}\, n!\,\zeta(m+1), \quad m=1,2,3\cdots$$

With Γ notation, we have

$$\frac{d^{n+1}}{dz^{n+1}}\ln\Gamma(z) = \frac{d^n}{dz^n}\psi(z) = \psi^{(n)}(z)$$

· Maclarin Expansion, Computation

It is now possible to write a Maclarin expansion for $\ln\Gamma(z+1)$:

$$\ln\Gamma(z+1) = \sum_{n=1}^{\infty}\frac{z^n}{n!}\left[\ln\Gamma(z+1)\right]^{(n)} = \sum_{n=1}^{\infty}\frac{z^n}{n!}\psi^{(n-1)}(1)$$

$$= -\gamma z + \sum_{n=2}^{\infty}\underbrace{(-1)^n (n-1)!\,\zeta(n)}_{\psi^{(n-1)}(1)}\frac{z^n}{n!}$$

$$= -\gamma z + \sum_{n=2}^{\infty}(-1)^n\frac{z^n}{n}\zeta(n)$$

convergent for $|z|<1$. For $z=x$, the range is $-1<x\leq 1$

Example 8.2.1 Catalan's constant

Catalan's constant, or $\beta(2)$ is given by

$$K = \beta(2) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$$

$$K = 1 - \frac{1}{9} + \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k+1)^2} \qquad \text{separating } k=2n \text{ and } k=2n+1 \text{ terms}$$

$$= 1 - \frac{1}{9} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{(4n+1)^2}}_{k=2n} - \underbrace{\sum_{n=1}^{\infty} \frac{1}{(4n+3)^2}}_{k=2n+1}$$

Since $\psi^{(m)}(z+1) = (-1)^{m+1} m! \sum_{n=1}^{\infty} \frac{1}{(z+n)^{m+1}}$,

and $\sum_{n=1}^{\infty} \frac{1}{(4n+1)^2} = \frac{1}{16} \sum_{n=1}^{\infty} \frac{1}{(n+\frac{1}{4})^2}$, $\psi^{(1)}(\frac{1}{4}+1) = \sum_{n=1}^{\infty} \frac{1}{(\frac{1}{4}+n)^2}$

we get $K = \frac{8}{9} + \frac{1}{16}\psi^{(1)}(1+\frac{1}{4}) - \frac{1}{16}\psi^{(1)}(1+\frac{3}{4})$

8.3 Stiring's Series

From ch5, we have the Euler - Maclaurin Integration formula:

$$\int_0^n f(x)\,dx = \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)$$

$$- \sum_{p=1}^{q} \frac{1}{(2p)!} B_{2p} \left[f^{(2p-1)}(n) - f^{(2p-1)}(0) \right]$$

$$+ \frac{1}{(2q)!} \int_0^1 B_{2q}(x) \sum_{\nu=0}^{n-1} f^{(2q)}(x+\nu)\,dx$$

we define $(2n)!\, b_{2n} = B_{2n}$, and let $q \to \infty$,

$$\int_0^n f(x)\,dx = \frac{1}{2}f(0) + f(1) + f(2) + \cdots + f(n-1) + \frac{1}{2}f(n)$$

$$- b_2\left[f'(n) - f'(0)\right] - b_4\left[f'''(n) - f'''(0)\right] - \cdots$$

(the 3rd term $\frac{1}{(2q)!}\int_0^1 B_{2q} \sum f^{(2q)}(x+\nu)\,dx$ disappears)

Let $f(x) = \frac{1}{(z+x)^2}$ then $\int_0^{\infty} \frac{dx}{(z+x)^2} = -\frac{1}{(z+x)}\Big|_0^{\infty} = \frac{1}{z}$

Let $n \to \infty$ in the Euler-Maclaurin formula,

$$\int_0^\infty \frac{dx}{(z+x)^2} = \frac{1}{2}f(0) + f(1) + f(2) + \cdots$$

$$- b_2\,[f'(n) - f'(0)] - b_4\,[f'''(n) - f'''(0)]$$

$$f'(x) = -\frac{2!}{(z+x)^3} \qquad f'''(x) = -\frac{4!}{(z+x)^5}$$

$$f'(n) - f'(0) = 2!\left[-\frac{1}{(z+n)^3} + \frac{1}{z^3}\right] = 2!\,\frac{1}{z^3} \quad \text{as } n \to \infty$$

Therefore,

$$\frac{1}{z} = \underset{\uparrow\ \frac{1}{2}f(0)}{\frac{1}{2z^2}} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \frac{1}{(z+3)^2} + \cdots - \frac{2!\,b_2}{z^3} - \frac{4!\,b_4}{z^5} - \cdots$$

$$= \frac{1}{2z^2} + \psi^{(1)}(z+1) - \frac{2!\,b_2}{z^3} - \frac{4!\,b_4}{z^5} - \cdots \qquad \left(\because\ \psi^{(m)}(z+1) = (-1)^{m+1} m! \sum \frac{1}{(z+n)^{m+1}}\right)$$

Solving for $\psi^{(1)}(z+1)$, we get

$$\psi^{(1)}(z+1) = \frac{1}{z} - \frac{1}{2z^2} + \frac{B_2}{z^3} + \frac{B_4}{z^5} + \cdots = \frac{1}{z} - \frac{1}{2z^2} + \sum_{n=1}^{\infty} \frac{B_{2n}}{z^{2n+1}}$$

(Since Bernoulli numbers diverge strongly, this series does not converge)

Integrating once, (another expression for digamma function)

$$\psi(z+1) = C_1 + \ln z + \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)\,z^{2n}}$$

Integrating above, to z from $z-1$, LHS becomes

$$\int_{z-1}^{z} \psi(z'+1)\,dz' = \int_{z-1}^{z} \frac{d}{dz'} \ln\Gamma(z'+1)\,dz' = \ln\Gamma(z+1)\Big|_{z-1}^{z} = \ln\frac{\Gamma(z+1)}{\Gamma(z)}$$

$$\text{RHS} = C_1 + (z\ln z - z)\Big|_{z-1}^{z} + \frac{1}{2}\ln z\Big|_{z-1}^{z} + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)(2n-1)}\,\frac{1}{z^{(2n-1)}}\Big|_{z-1}^{z}$$

$$= C_1 + z\ln z - \cancel{z} - (z-1)\ln(z-1) + \cancel{z} - 1 + \frac{1}{2}\ln z - \frac{1}{2}\ln(z-1)$$

$$+ \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)(2n-1)} \left[\frac{1}{z^{2n-1}} - \frac{1}{(z-1)^{2n-1}}\right]$$

$$= C_1 + \left(z+\frac{1}{2}\right)\ln z - \left(z-\frac{1}{2}\right)\ln(z-1) - 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)(2n-1)}\left[1 - \frac{1}{(1-\frac{1}{z})^{2n-1}}\right]\frac{1}{z^{2n-1}}$$

$$= C_1 + \ln\frac{z^{z+\frac{1}{2}}}{(z-1)^{z-\frac{1}{2}}} - 1 + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)(2n-1)}\,\frac{1}{z^{2n-1}}\left[1 - \frac{1}{(1-\frac{1}{z})^{2n-1}}\right]$$

Letting $z \to \infty$,

$$\lim_{z\to\infty} \ln \frac{\Gamma(z+1)}{\Gamma(z)} = 0$$

$$\lim_{z\to\infty} \left(\ln \frac{(z)^{z+\frac{1}{2}}}{(z-1)^{z-1/2}} - 1 \right) = 0$$

$$\lim_{z\to\infty} \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)(2n-1)} \frac{1}{z^{2n-1}} \left[1 - \frac{1}{(1-1/z)^{2n-1}} \right] = 0$$

Therefore $C_1 = 0$ and we get

$$\psi(z+1) = \ln z + \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n z^{2n}}$$

- Stirling's Series

The indefinite integral of the digamma function is

$$\ln\Gamma(z+1) = C_2 + z\ln z - z + \frac{1}{2}\ln z + \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} \frac{1}{z^{(2n-1)}}$$

$$= C_2 + \left(z+\tfrac{1}{2}\right)\ln z - z + \frac{B_2}{2z} + \cdots + \frac{B_{2n}}{2n(2n-1)\,z^{(2n-1)}} + \cdots$$

Let's assume (will get in Section 8.4)

$$\Gamma(z+1)\,\Gamma\left(z+\tfrac{1}{2}\right) = 2^{-2z}\sqrt{\pi}\,\Gamma(2z+1)$$

: doubling formula or Legendre duplication formula

taking a log on the formula, we get

$$\ln\Gamma(z+1) + \ln\Gamma\left(z+\tfrac{1}{2}\right) = -2z\ln 2 + \tfrac{1}{2}\ln\pi + \ln\Gamma(2z+1)$$

$$\ln\Gamma\left(z+\tfrac{1}{2}\right) = C_2 + z\ln\left(z-\tfrac{1}{2}\right) - \left(z-\tfrac{1}{2}\right) + \frac{B_2}{2(z-\frac{1}{2})} + \cdots$$

$$\ln\Gamma(2z+1) = C_2 + \left(2z+\tfrac{1}{2}\right)\ln(2z) - 2z + \frac{B_2}{4z} + \cdots$$

comparing constant terms,

$$C_2 = \tfrac{1}{2}\ln\pi + \tfrac{1}{2}\ln 2 = \tfrac{1}{2}\ln 2\pi$$

So,

$$\ln\Gamma(z+1) = \frac{1}{2}\ln 2\pi + \left(z+\frac{1}{2}\right)\ln z - z + \frac{1}{12z} - \frac{1}{360z^3} + \frac{1}{1260z^5} - \cdots$$

: Stirling's series

8.4. The Beta Function

We know that (eq. 8.25)

$$z! \equiv \int_0^\infty e^{-t} t^z \, dt \qquad Re(z) > -1$$

Now, we write the product of two factorials as the product of two integrals (we take the integrals over a finite range)

$$m!n! = \lim_{a^2\to\infty} \int_0^{a^2} e^{-u} u^m \, du \int_0^{a^2} e^{-v} v^n \, dv \qquad \begin{matrix} Re(m) > -1 \\ Re(n) > -1 \end{matrix}$$

Replacing u with x^2 and v with y^2,

$$du = 2x\,dx \; , \; dv = 2y\,dy,$$

$$m!n! = \lim_{a\to\infty} 4 \int_0^a e^{-x^2} x^{2m+1} \, dx \int_0^a e^{-y^2} y^{2n+1} \, dy$$

Now, $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$ gives

$$m!n! = \lim_{a\to\infty} 4 \int_0^a e^{-r^2} r^{2m+2n+2} \, r\,dr \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta \, d\theta$$

$$= \lim_{a\to\infty} 4 \int_0^a e^{-r^2} r^{2m+2n+3} \, dr \int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta \, d\theta$$

If we assume (Exer. 8.1.11)

$$\int_0^\infty x^{2s+1} e^{-ax^2} dx = \frac{s!}{2a^{s+1}} \; ,$$

The first term becomes

$$\int_0^a r^{2m+2n+3} e^{-r^2} dr = \int_0^a r^{2(m+n+1)+1} e^{-r^2} dr = \frac{(m+n+1)!}{2}$$

$$\therefore \; m!n! = (m+n+1)! \; 2 \int_0^{\pi/2} \cos^{2m+1}\theta \cdot \sin^{2n+1}\theta \, d\theta$$

We define the beta function as:

$$B(m+1, n+1) \equiv 2 \int_0^{\pi/2} \cos^{2m+1}\theta \cdot \sin^{2n+1}\theta \, d\theta$$

$$= \frac{m!\,n!}{(m+n+1)!}$$

Equivalently, we write $B(p,q) = \dfrac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$, $B(q,p) = B(p,q)$

- Definite Integrals, Alternate Forms

If we put $t = \cos^2\theta$, $\quad dt = 2\cos\theta \cdot (-\sin\theta)\, d\theta$

$$B(m+1, n+1) = 2\int_0^{\pi/2} \cos^{2m+1}\theta \, \sin^{2n+1}\theta \cdot d\theta$$

$$= 2\int_1^0 \cos^{2m+1}\theta \, \sin^{2n+1}\theta \;\; \frac{1}{2}\cdot(-1)\cos^{-1}\theta \, \sin^{-1}\theta \; dt$$

$$= \int_0^1 t^m (1-t)^n \, dt$$

Replacing t by x^2, we get $\quad (\, dt = 2x\, dx \,)$

$$B(m+1, n+1) = \int_0^1 x^{2m} (1-x^2)^n \cdot 2x \, dx$$

$$\rightarrow \frac{m!\,n!}{2(m+n+1)!} = \int_0^1 x^{2m+1} (1-x^2)^n \, dx$$

The substitution $t = \frac{u}{1+u}$ yields $\quad dt = \frac{1+u-u}{(1+u)^2}du = \frac{du}{(1+u)^2}$, $1-t = \frac{1+u-u}{1+u} = \frac{1}{1+u}$

$$B(m+1, n+1) = \int_0^\infty \left(\frac{u}{1+u}\right)^m \left(\frac{1}{1+u}\right)^n \frac{du}{(1+u)^2}$$

$$= \int_0^\infty \frac{u^m}{(1+u)^{m+n+2}} \cdot du$$

- Verification of $\pi\alpha / \sin\pi\alpha$ relation

If we take $\quad m = a, \; n = -a, \; -1 < a < 1$

$$a!(-a)! = \int_0^\infty \frac{u^a}{(1+u)^2} \, du = \frac{\pi a}{\sin\pi a}$$

(see Eq. 8.23 and below)

- Derivation of Legendre Duplication Formula

From $\dfrac{m!\,n!}{(m+n+1)!} = \displaystyle\int_0^1 t^m (1-t)^n \, dt$,

if we set $m = n = z$, and $Re(z) > -1$

$$\frac{z!\,z!}{(2z+1)!} = \int_0^1 t^z (1-t)^z \, dt$$

By substituting $t = (1+s)/2$, we have $\quad (\, dt = \frac{ds}{2}, \; 1-t = \frac{1-s}{2} \,)$

$$\frac{z!\,z!}{(2z+1)!} = \int_{-1}^1 \left(\frac{1+s}{2}\right)^z \left(\frac{1-s}{2}\right)^z \frac{ds}{2} = 2^{-2z-1} = 2^{-2z} \int_0^1 (1-s^2)^z \, ds$$

Since we have $\dfrac{m!\,n!}{2(m+n+1)!} = \int_0^1 x^{2m+1}(1-x^2)^n \, dx$,

by having $m = -\frac{1}{2}$, $n = z$, we have

$$\frac{(-\frac{1}{2})!\, z!}{2\,(z+\frac{1}{2})!} = \int_0^1 (1-x^2)^z \, dx$$

$$\therefore \quad \frac{z!\,z!}{(2z+1)!} = 2^{-2z-1} \frac{z!\,(-\frac{1}{2})!}{(z+\frac{1}{2})!}$$

with $(-\frac{1}{2})! = \sqrt{\pi}$,

$$z!\,(z+\tfrac{1}{2})! = 2^{-2z-1}\sqrt{\pi}\,(2z+1)!$$: Legendre duplication formula

Dividing by $(z+\frac{1}{2})$,

$$z!\,(z-\tfrac{1}{2})! = 2^{-2z-1}\sqrt{\pi}\,(2z+1)! \,/\, (z+\tfrac{1}{2})$$

$$= 2^{-2z}\sqrt{\pi}\,(2z)!$$

Even if we restrict $Re(z) > -1$, (it holds for all regular points z by analytic continuation)

From $z!\,(z+\frac{1}{2})! = 2^{-2z-1}\sqrt{\pi}\,(2z+1)!$, by letting $z = n$,

$$(n+\tfrac{1}{2})! = 2^{-2n-1}\sqrt{\pi}\,\frac{(2n+1)!}{n!}$$

since $(2n+1)!! = \dfrac{(2n+1)!}{2^n n!}$,

$$(n+\tfrac{1}{2})! = 2^{-2n-1}\sqrt{\pi}\,\frac{(2n+1)!}{n!}\,\frac{2^n}{2^n} = \sqrt{\pi}\,\frac{(2n+1)!!}{2^{n+1}}$$

- Incomplete Beta Function

The Beta function was $B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$

we define an incomplete beta function as

$$B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\,dt \qquad (0 \le x \le 1,\ p>0,\ q>0)$$

: appears in probability theory

8.5 The incomplete gamma function and related functions

we define the incomplete gamma functions by:

$$\gamma(a,x) \equiv \int_0^x e^{-t}\, t^{a-1}\, dt \qquad Re(a)>0$$

and $$\Gamma(a,x) = \int_x^\infty e^{-t}\, t^{a-1}\, dt$$

clearly, $$\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$$

If the parameter a is a positive integer $(a=n)$

$$\gamma(n,x) = \int_0^x e^{-t} t^{n-1}\, dt$$

$$= \left(-e^{-t} t^{n-1}\right)\Big|_0^x + (n-1)\int_0^x e^{-t} t^{n-2}\, dt$$

$$= -e^{-x} x^{n-1} + (n-1)\int_0^x e^{-t} t^{n-2}\, dt \qquad \text{induction(?)}$$

$$= -e^{-x} - (n-1)e^{-x}x^{n-2} - (n-1)(n-2)\, e^{-x} x^{n-3} - \cdots - (n-1)(n-2)\cdots 2e^{-x}\cdot x + (n-1)(n-2)\cdots 1\cdot \underbrace{\int_0^x e^{-t}\, dt}_{-e^{-x}+1}$$

$$= -e^{-x} - (n-1)e^{-x}x^{n-2} - (n-1)(n-2)e^{-x}x^{n-3} - \cdots - (n-1)(n-2)\cdots 2e^{-x}\cdot x - (n-1)(n-2)\cdots 2\cdot 1\, e^{-x} + (n-1)!$$

$$= (n-1)!\left[1 - e^{-x} - e^{-x}\cdot x - \frac{e^{-x}}{2}x^2 - \frac{e^{-x}}{2\cdot 3}x^3 - \cdots \frac{-e^{-x}}{(n-1)!}x^{n-1}\right]$$

$$= (n-1)!\left(1 - \sum_{s=0}^{n-1} \frac{x^s}{s!}\right)$$

Similarly, one can show that

$$\Gamma(n,x) = (n-1)!\, e^{-x} \sum_{s=0}^{n-1} \frac{x^s}{s!} \qquad n=1,2\ldots$$

For nonintegral a,

$$\gamma(a,x) = \int_0^x e^{-t}\, t^{a-1}\, dt = \int_0^x \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n\, t^{a-1}\, dt$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^x t^{n+a-1}\, dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!}\, \frac{x^{n+a}}{n+a}$$

$$= x^a \sum_{n=0}^\infty (-1)^n \frac{x^n}{n!(a+n)} \qquad |x| \sim 1 \quad (\text{small } x)$$

Similarly, one can show that (Ex. 5.7.7)

$$\Gamma(a,x) = x^{a-1} e^{-x} \sum_{n=0}^{\infty} \frac{(a-1)!}{(a-1-n)!} \cdot \frac{1}{x^n}$$

Now, $(a-1)! = \Gamma(a)$, $(a-1-n)! = \Gamma(a-n) = \Gamma(1-(n-a+1))$

With $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$,

$$\frac{(a-1)!}{(a-1-n)!} = \frac{\Gamma(a)}{\Gamma(a-n)} = \frac{1}{\Gamma(1-a)} \frac{\pi}{\sin a\pi} \cdot \frac{\sin(n-a+1)\pi z}{\pi} \Gamma(n-a+1)$$

Note : $\sin(n-a+1)\pi z = (-1)^n \sin a\pi z$

$$\frac{(a-1)!}{(a-1-n)!} = \frac{\Gamma(n-a+1)}{\Gamma(1-a)} (-1)^n = (-1)^n \frac{(n-a)!}{(-a)!}$$

$$\therefore \Gamma(a,x) = x^{a-1} e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{(n-a)!}{(-a)!} \cdot \frac{1}{x^n}, \quad x \gg 1$$

: asymptotic expansion of $\Gamma(a,x)$

· Exponential Integral

We define the exponential integral by

$$E_1(x) \equiv \int_x^{\infty} \frac{e^{-t}}{t} dt = -Ei(-x)$$

: the above diverges logarithmically as $x \to 0$

To obtain a series expansion for small x,

$$E_1(x) = \int_x^{\infty} e^{-t} t^{-1} dt = \Gamma(0,x)$$

$$= \lim_{a\to 0} \left[\int_0^{\infty} e^{-t} t^{a-1} dt - \int_0^{x} e^{-t} t^{a-1} dt \right]$$

$$= \lim_{a\to 0} \left[\Gamma(a) - \gamma(a,x) \right]$$

Since $\gamma(a,x) = x^a \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!(a+n)}$ (page 15 in this note)

$$E_1(x) = \lim_{a\to 0} \left[\Gamma(a) - x^a \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!(a+n)} \right]$$

$$= \lim_{a\to 0} \left[\frac{a\Gamma(a) - x^a}{a} \right] - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}$$

Now, $\lim_{a\to 0} \dfrac{a\Gamma(a) - x^a}{a} = \lim_{a\to 0} \dfrac{\frac{d}{da}[a\Gamma(a) - x^a]}{\frac{d}{da}(a)}$ (l'Hopital's rule)

and since

$$\frac{d}{da}(a\Gamma(a)) = \frac{d}{da}(a!) = \frac{d}{da}(e^{\ln a!}) = a!\frac{d}{da}(\ln a!)$$
$$= a!\,\psi(a+1)$$

where $\psi(z+1) \equiv \frac{d}{dz}[\ln\Gamma(z+1)]$, $\psi(1) = -\gamma$

$$\frac{d}{da}(x^a) = \frac{d}{da}(e^{\ln x^a}) = x^a \cdot \frac{d}{da}(a\ln x) = x^a \cdot \ln x$$

So, $E_1(x) = \lim_{a\to 0}\left[a!\,\psi(a+1) - x^a \cdot \ln x\right] - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}$

$$= -\gamma - \ln x - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n \cdot n!}$$

we obtain the rapidly converging series

(diverges logarithmically as $x \to 0$)

Further special forms: (sine integral, cosine integral, logarithmic integral)

$$si(x) = -\int_x^{\infty} \frac{\sin t}{t}\,dt$$

$$Ci(x) = -\int_x^{\infty} \frac{\cos t}{t}\,dt$$

$$li(x) = \int_0^x \frac{du}{\ln u} = Ei(\ln\ )$$

We can show that

$$si(x) = \frac{1}{2i}\left[Ei(x) - Ei(-ix)\right] = \frac{1}{2i}\left[E_1(ix) - E_1(-ix)\right]$$

$$Ci(x) = \frac{1}{2}\left[Ei(ix) + Ei(-ix)\right] = -\frac{1}{2}\left[E_1(ix) + E_1(-ix)\right]$$

$$Ei(ix) = Ci(x) + i\,si(x)$$
$$E_1(ix) = -Ci(x) + i\,si(x)$$

Error Integral

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \qquad \operatorname{erfc} z = 1 - \operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt$$