Ch 4 Group theory 4.1 Introduction of Group Theory

In classical mechanics, we learned that

symmetry of a physical system leads to conservation laws

ex) rotational symmetry $\longleftrightarrow$ conservation of angular momentum

: has to do with mathematical structure called group

[discrete group (finite)

[continuous " (infinite)

· Definition

A set of objects or operations G with

1. If $a \in G$ and $b \in G$ then $ab \in G$;

$(a,b) \to ab$ maps $G \times G$ onto G

2. Multiplication is associative $(ab)c = a(bc)$

3. $\exists\ I \in G$ such that $Ia = aI = a \quad \ni \forall\ a \in G$

4. $\exists\ a^{-1} \in G$ " $a^{-1}a = aa^{-1} = I$ "

ex) a set of counterclockwise coordinate rotations : G

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$$

1. $$R(\phi_1)R(\phi_2) = \begin{pmatrix} \cos\phi_1 & \sin\phi_1 \\ -\sin\phi_1 & \cos\phi_1 \end{pmatrix}\begin{pmatrix} \cos\phi_2 & \sin\phi_2 \\ -\sin\phi_2 & \cos\phi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 & \cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2 \\ -\sin\phi_1\cos\phi_2 - \cos\phi_1\sin\phi_2 & -\sin\phi_1\sin\phi_2 + \cos\phi_1\cos\phi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\phi_1+\phi_2) & \sin(\phi_1+\phi_2) \\ -\sin(\phi_1+\phi_2) & \cos(\phi_1+\phi_2) \end{pmatrix} = R(\phi_1+\phi_2) \in G$$

2. $\left(R(\phi_1)R(\phi_2)\right)R(\phi_3) = R(\phi_1)\left(R(\phi_2)R(\phi_3)\right)$

If we define $\quad \because$ matrix multiplication is associative

3. $\forall\ I = R(\phi=0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\quad IR = RI = R \in G$

4. $R^{-1}(\phi) = R(-\phi) \qquad R(-\phi)R(\phi) = R(\phi)R(-\phi) = I \quad \ni \forall\ R(\phi) \in G$

Further more,

$R(\phi_1)R(\phi_2) = R(\phi_2)R(\phi_1)$: we call this kind of group as commutative or Abelian group

If $G' \subset G$ and G' forms a group itself, G' is called subgroup of G

ex) $G: R(\phi)$ and $G' = \{ I, R(\phi=\pi) \}$

If $gg'g^{-1} \in G'$ $\exists^{\forall}$ $g \in G$ and $g' \in G'$,

G' is said to be an invariant subgroup of G

orthogonal $n \times n$ matrices form a group : $O(n)$

$O(n)$ with $\det(O(n)) = +1$: $SO(n)$

ex) $R(\phi)$ above forms a group $SO(2)$

1. If $O_1 \in SO(n)$ and $O_2 \in SO(n)$

$$\left.\begin{array}{l} \widetilde{O_1 O_2} = \tilde{O}_2 \tilde{O}_1 = O_2^{-1} O_1^{-1} = (O_1 O_2)^{-1} \\ \det(O_1 O_2) = \det(O_1)\det(O_2) = +1 \end{array}\right] \quad O_1 O_2 \in SO(n)$$

2. $(O_1 O_2) O_3 = O_1 (O_2 O_3)$
3. $I_n O_i = O_i I \quad \exists\, {}^{\forall} O_i \in SO(n)$
4. $O_i^{-1} = \tilde{O}_i$ from the orthogonality condition

Real orthogonal $n \times n$ matrix has $\frac{n(n-1)}{2}$ independent parameters

Γ orthogonality condition is

$$\sum_i^n a_{ij} a_{ik} = \delta_{jk}$$

\# of a_{ij}'s : $n \times n = n^2$

\# of constraints from the orthogonality condition

: # of independent pairs of (j,k) = upper half triangle including diagonal term

$= (n+1)\frac{n}{2}$ $(= n + (n-1) + \cdots + 1)$

$\therefore$ # of independent parameters $= n^2 - \frac{n(n+1)}{2}$

$$= \frac{n(n-1)}{2}$$

unitary $n \times n$ matrices form the group $U(n)$

$U(n)$ with $\det(U(n)) = +1$: $SU(n)$

· Homomorphism and Isomorphism

There may be a correspondance between the elements of two groups : one-to-one, two-to-one or many-to-one

Two groups are homomorphic if group multiplication is preserved

ex) $SO(3)$ and $SU(2)$

If the correspondence is one-to-one then two groups are isomorphic

ex) $R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}$ and $R_z(\phi) \equiv \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The group of rotations R_z is obviously isomorphic to the group of rotations $R(\phi)$

· Matrix Representations - Reducible and Irreducible

Assume

$$H\psi = E\psi$$

Stays invariant under a group G of transformations R (in G) that is $H_R = RHR^{-1} = H$.

Γ: Multiplying R in $H\psi = E\psi$,

$$RH\psi = RE\psi = ER\psi$$
$$= (RHR^{-1})R\psi$$
$$= HR\psi \qquad \rightarrow H(R\psi) = E(R\psi) \lrcorner$$

In other words, $R\psi$ has the same energy E (called multiplet)

Let us assume that this vector space V_ψ of transformed solutions has a finite dimension n.

Let $\psi_1 \psi_2 \cdots \psi_n$ be a basis. Because $R\psi_j$ is a member of the multiplet, we can expand it as

$$R\psi_j = \sum_k r_{jk} \psi_k$$

Thus, with each R in G we can associate a matrix (r_{jk}), and this map $R \rightarrow (r_{jk})$ is called a representation of G

If we can take any element of V_ϕ and by rotating with all element R of G transform it into all other elements of V_ϕ

: the representation is irreducible

If all elements of V_ϕ are not reached, then V_ϕ splits it into a direct sum of two or more vectorsubspaces $V_\phi = V_1 \oplus V_2 \oplus \cdots$

: the representation is reducible

If (r_{jk}) is reducible, then we can find a basis in V_ϕ (or there is a unitary matrix U) so that

$$U (r_{jk}) U^{-1} = \begin{pmatrix} \vec{r}_1 & 0 & 0 \cdots \\ 0 & \vec{r}_2 & \\ 0 & 0 & \\ \vdots & \vdots & \end{pmatrix}$$

for all R of G and all matrices (r_{jk}) Here $\vec{r}_1$, $\vec{r}_2$ are matrices of lower dimension than (r_{jk}) that are lined up along the diagonal.

4.2 Generators of Continuous Group

Let us consider $SO(2)$ group as a simple example:

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma_2^2 = I_2$

$$R(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} = I_2 \cos\phi + i\sigma_2 \sin\phi$$

$$= I_2 \left(1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \dots \right) + i\sigma_2 \left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \dots \right)$$

$$= \left(I_2 + \frac{(i\sigma_2\phi)^2}{2!} + \frac{(i\sigma_2\phi)^4}{4!} + \dots \right) + \left(i\sigma_2\phi + \frac{(i\sigma_2\phi_3)^3}{3!} + \frac{(i\sigma_2\phi_3)^5}{5!} + \dots \right)$$

$$= I_2 + (i\sigma_2\phi) + \frac{(i\sigma_2\phi)^2}{2!} + \frac{(i\sigma_2\phi)^3}{3!} + \dots$$

$$= e^{i\sigma_2\phi}$$

$$R(\phi_2)R(\phi_1) = e^{i\sigma_2\phi_2} e^{i\sigma_2\phi_1} = e^{i\sigma_2(\phi_1+\phi_2)} = R(\phi_1+\phi_2)$$

Now, we look for an exponential representation

$$R = e^{i\epsilon S} \quad \text{when } \epsilon \to 0$$

S: generators of group G ex) σ_2 for $SO(2)$

S form a linear space whose dimension is the order of G

If R does not change the volume element, $\det(R) = 1$

$$\det(R) = \det(e^{i\epsilon S}) = e^{i\epsilon \, \mathrm{tr} S} = 1 \quad \text{so,} \quad \mathrm{tr}(S) = 0$$

If R in G is unitary $(R^\dagger = R^{-1})$, $\quad RR^\dagger = 1 = e^{i\epsilon S} e^{-i\epsilon S^\dagger} = e^{i\epsilon(S - S^\dagger)}$

$\therefore S = S^\dagger$ so S is Hermitian

We expand the group elements in Taylor series:

$$R_i = e^{i\epsilon_i S_i} = 1 + i\epsilon_i S_i - \frac{1}{2}\epsilon_i^2 S_i^2 + \dots$$

$$R_i = e^{-i\epsilon_i S_i} = 1 - i\epsilon_i S_i - \frac{1}{2}\epsilon_i^2 S_i^2 + \dots$$

If we do

$$R_i^{-1} R_j^{-1} R_i R_j = (1 - i\epsilon_i S_i - \tfrac{1}{2}\epsilon_i^2 S_i^2 + \cdots)(1 - i\epsilon_j S_j - \tfrac{1}{2}\epsilon_j^2 S_j^2 + \cdots)$$
$$(1 + i\epsilon_i S_i - \tfrac{1}{2}\epsilon_i^2 S_i^2 + \cdots)(1 + i\epsilon_j S_j - \tfrac{1}{2}\epsilon_j^2 S_j^2 + \cdots)$$
$$= 1 + \epsilon_i\epsilon_j(-S_iS_j + S_iS_j + S_jS_i - S_iS_j)$$
$$\epsilon_i\epsilon_i(S_iS_i) + \epsilon_j\epsilon_j(S_jS_j) - \epsilon_i^2 S_i^2 - \epsilon_j^2 S_j^2$$
$$+ O(\epsilon^3) \text{ or higher}$$
$$= 1 + \epsilon_i\epsilon_j [S_j, S_i] + \cdots$$

Since $R_i^{-1} R_j^{-1} R_i R_j = R_{ij} \in G$ and close to unity,

we assume

$$R_{ij} = 1 + \epsilon_i\epsilon_j \sum_k C_{ji}^k S_k$$

so we get the closure relation of the generators of the Lie group G:

$$[S_i, S_j] = \sum_k C_{ij}^k S_k$$

The coefficients C_{ij}^k are called the structure constants of the group G.

Because the commutator is antisymmetric in i and j,

$$C_{ij}^k = -C_{ji}^k$$

If the commutator in $[S_i, S_j] = \sum_k C_{ij}^k S_k$ is taken as a multiplication law of generators, the vector space of generators becomes an algebra, the Lie algebra $\mathcal{G}$ of the group G

Jacobi identity:

$$[[S_i, S_j], S_k] + [[S_j, S_k], S_i] + [[S_k, S_i], S_j] = 0$$

$$\Gamma = [(S_iS_j - S_jS_i), S_k] + [(S_jS_k - S_kS_j), S_i] + [(S_kS_i - S_iS_k), S_j]$$
$$= S_iS_jS_k - S_jS_iS_k - S_kS_iS_j + S_kS_jS_i$$
$$+ S_jS_kS_i - S_kS_jS_i - S_iS_jS_k + S_iS_kS_j$$
$$+ S_kS_iS_j - S_iS_kS_j - S_jS_kS_i + S_jS_iS_k$$
$$= 0$$

If we use the closure relation to Jacobi identity:

$$\to \left[\sum_m c_{ij}^m S_m , S_k \right] + \left[\sum_m c_{jk}^m S_m , S_i \right] + \left[\sum_m c_{ki}^m S_m , S_j \right] = 0$$

$$= \sum_m \left\{ c_{ij}^m [S_m , S_k] + c_{jk}^m [S_m , S_i] + c_{ki}^m [S_m , S_j] \right\}$$

$$= \sum_{mn} \left\{ c_{ij}^m c_{mk}^n S_n + c_{jk}^m c_{mi}^n S_n + c_{ki}^m c_{mj}^n S_n \right\} = 0$$

$$\therefore \quad \sum_{mn} \left\{ c_{ij}^m c_{mk}^n + c_{jk}^m c_{mi}^n + c_{ki}^m c_{mj}^n \right\} = 0$$

For SO(2), there is only one linearly independent generator

$$\left(\frac{n(n-1)}{2} = \frac{2}{2}(2-1) = 1 \right)$$

we get the generator by differentiation at the unity of SO(2):

$$(-i)\left.\frac{dR(\phi)}{d\phi}\right|_{\phi=0} = (-i)\left.\begin{pmatrix} -\sin\phi & \cos\phi \\ -\cos\phi & -\sin\phi \end{pmatrix}\right|_{\phi=0} = (-i)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$$

For $R_z(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(-i)\left.\frac{dR(\phi)}{d\phi}\right|_{\phi=0} = S_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The rotation $R_z(\delta\phi)$ through an infinitesimal angle $\delta\phi$ may then be written as

$$R_z(\delta\phi) \simeq I_3 + i\,\delta\phi\, S_z$$

A finite rotation $R(\phi)$ may be composed of successive infinitesimal rotations : $\delta\phi = \frac{\phi}{N}$ for N rotations

$$R_z(\phi) = \lim_{N\to\infty}\left[1 + \frac{i\phi}{N}S_z\right]^N = \lim_{N\to\infty} \underbrace{\left(1+\frac{i\phi}{N}S_z\right)\left(1+\frac{i\phi}{N}S_z\right)\cdots\left(1+\frac{i\phi}{N}S_z\right)}_{N \text{ times}}$$

$$= 1 + i\phi S_z + \frac{(i\phi S_z)^2}{2!} + \frac{(i\phi S_z)^3}{3!} + \dots$$

$$= e^{i\phi S_z}$$

: this form identifies S_z as the generator of the group R_z, an Abelian subgroup of SO(3)

From

$$R_x(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix} \qquad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

we get generators

$$S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad S_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

of R_x (R_y), the subgroup of rotations about the x-(y-) axis.

Rotation of Functions and Orbital Angular Momentum

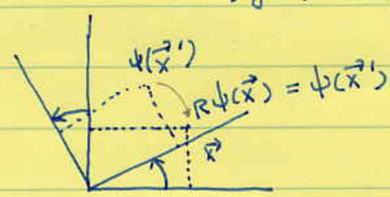
So far, the group elements are matrices that rotate the coordinates.
Now, let us hold the coordinates fixed and rotate a function $\psi(x,y,z)$ relative to our fixed coordinates.

With R to rotate the coordinates,

$$\vec{x}\,' = R\vec{x}$$

we define R on ψ by

$$R\psi(x,y,z) = \psi'(x,y,z) \equiv \psi(\vec{x}\,')$$



If R rotates the coordinates counterclockwise, the effect of R is to rotate the pattern of the ψ clockwise ⊛

From $x' = x\cos\phi + y\sin\phi$, $y' = -x\sin\phi + y\cos\phi$, for infinitesimal transformation, we get

$$x' \sim x + y\,\delta\phi, \quad y' = y - x\,\delta\phi$$

Then

$$\begin{aligned}
R_z(\delta\phi)\psi(x,y,z) &= \psi(x + y\,\delta\phi,\ y - x\,\delta\phi,\ z) \\
&= \psi(x,y,z) + \frac{\partial\psi}{\partial x}\,y\,\delta\phi + \frac{\partial\psi}{\partial y}(-x\,\delta\phi) + \mathcal{O}(\delta\phi^2) \\
&= \psi(x,y,z) - \delta\phi\left\{x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x}\right\} + \mathcal{O}(\delta\phi^2) \\
&\equiv \left(1 - i\,\delta\phi\, L_z\right)\psi(x,y,z)
\end{aligned}$$

where

$$L_z = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) = i\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

L_z : z-component of the orbital angular momentum in Quantum Mechanics

$$R_z(\phi + \delta\phi)\psi = R_z(\delta\phi)R_z(\phi)\psi = (1 - i\,\delta\phi\, L_z)R_z(\phi)\psi$$

we have

$$\frac{dR_z}{d\phi} = \lim_{\delta\phi \to 0}\frac{R_z(\phi+\delta\phi) - R_z(\phi)}{\delta\phi} = \frac{-i\,\delta\phi\, L_z R_z(\phi)}{\delta\phi}$$

$$= -i L_z R_z(\phi)$$

$\delta\theta$ ⊗ clockwise ... 8

So, $R_z(\phi) = e^{-i\phi L_z}$ (with $R_z(0)=1$)

Note: $(x,y,z) S_z \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = (x\ y\ z) \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = (iy\ \ -ix\ \ 0) \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

$= L_z$

we can set $L_x = (x\ y\ z) S_x \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$ $\quad L_y = (x\ y\ z) S_y \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$

We can show that

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

- $SU(2) - SO(3)$ Homomorphisms

$SU(2)$: unitary 2×2 matrices with determinant +1

has three Pauli matrices σ_i as generators

order 3 and depends on 3 real continuous parameters

(ξ, η, ζ) : Caley-Klein parameters

Its general form:

$$U_2(\xi,\eta,\zeta) = \begin{pmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ -e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

The relation between the generators and Caley-Klein parameters:

$$-i \left.\frac{\partial U_2}{\partial \xi}\right|_{\xi=0,\eta=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$$

$$-i(\sin\eta)^{-1} \left.\frac{\partial U_2}{\partial \zeta}\right|_{\zeta=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

$$(-i) \left.\frac{\partial U_2}{\partial \eta}\right|_{\eta=0,\zeta=0} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$$

Elements U_1, U_2, U_3 of $SU(2)$ may be generated by

$U_1 = e^{\frac{i}{2}a_1\sigma_1}$ $\quad U_2 = e^{\frac{i}{2}a_2\sigma_2}$ $\quad U_3 = e^{\frac{i}{2}a_3\sigma_3}$

We put factor $\frac{1}{2}$ as

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = i\epsilon_{ijk}\frac{\sigma_k}{2}$$

we $S_i = \frac{1}{2}\sigma_i$ so

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

We have as the corresponding rotation operator in 2-dim. complex space $e^{i\phi\frac{\sigma_3}{2}} = R_z(\phi)$

So, for rotating 2-component vector wavefunction (spinor) or a spin ½ particle relative to fixed coordinates, the rotation operator is $R_z(\phi) = e^{-i\phi\frac{\sigma_3}{2}}$

Using the Euler identity, we have

$$U_j = \cos\left(\frac{a_j}{2}\right) + i\sigma_j \sin\left(\frac{a_j}{2}\right) \qquad (\because \sigma_j^2 = 1)$$

The general form of the $SU(2)$ matrix may be written as

$$U(\alpha\beta\gamma) = e^{-\frac{i\gamma\sigma_3}{2}} e^{-\frac{i}{2}\beta\sigma_2} e^{-\frac{i}{2}\alpha\sigma_3}$$

With the rotation matrix convension:

$$D(\alpha,\beta,\gamma) = U^{+}(\alpha,\beta,\gamma)$$

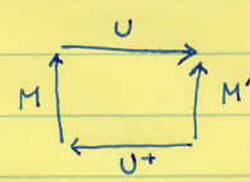
$$D^{1/2}(\alpha,\beta,\gamma) = e^{\frac{i\alpha\sigma_3}{2}} e^{\frac{i\beta\sigma_2}{2}} e^{\frac{i\gamma\sigma_3}{2}}$$

$$D^{L}(\alpha,\beta,\gamma) = e^{i\alpha L_z} e^{i\beta L_y} e^{i\gamma L_z}$$

$$D^{J}_{m'm}(\alpha,\beta,\gamma) = \langle Jm|\, e^{i\alpha J_z} e^{i\beta J_y} e^{i\gamma J_z} \,|Jm'\rangle$$

The operation of $SU(2)$ on a matrix is given by a unitary transformation

$$M' = UMU^{+}$$



Let M be the zero-trace matrix

$$M = x\sigma_1 + y\sigma_2 + z\sigma_3 = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$$

$$M' = x'\sigma_1 + y'\sigma_2 + z'\sigma_3 = \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix}$$