PHYS151 Lecture 04

Ch 04 Motion in Two and Three Dimensions

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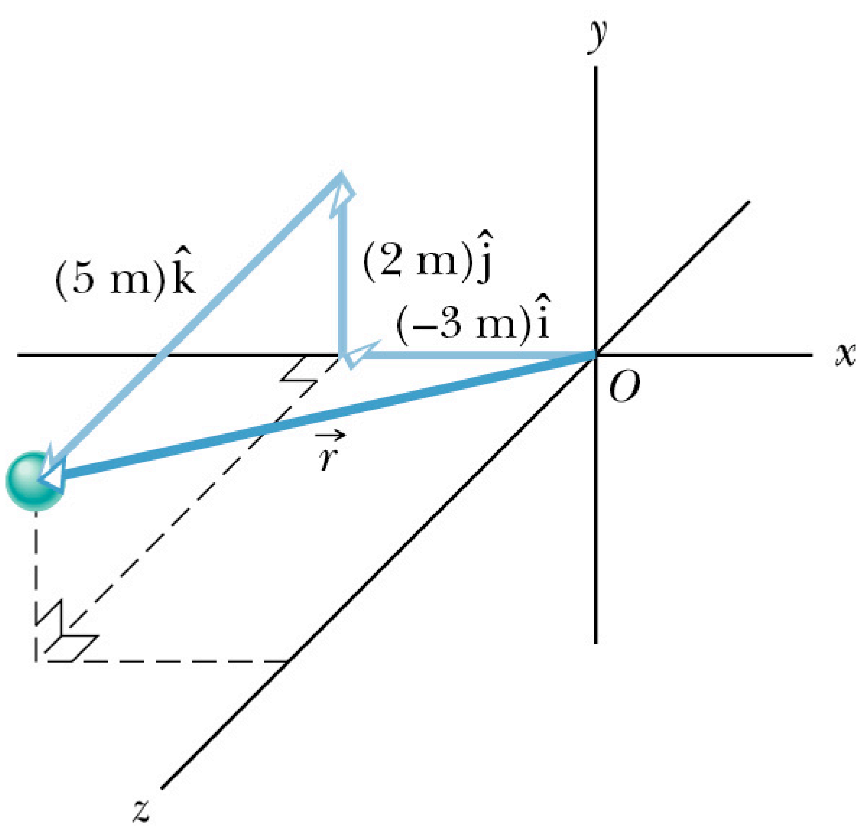
Position and Displacement

One general way of locating a particle:

position vector

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

ex) $\vec{r} = (\text{-3 m})\hat{i} + (2\text{ m}\hat{j}) + (5\text{ m})\hat{k}$



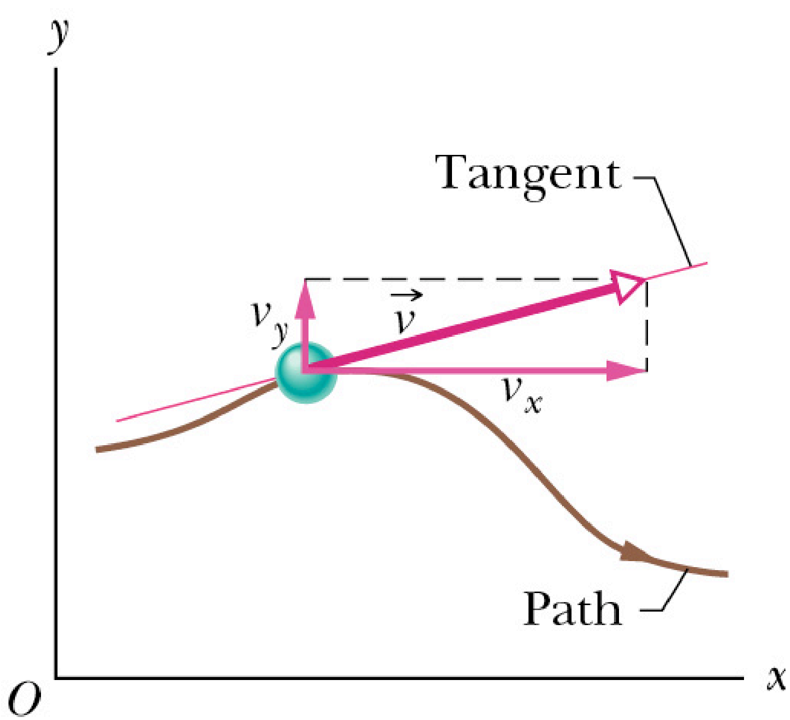
Displacement vector:

$$\begin{aligned}
\Delta\vec{r} &= \vec{r}_2 - \vec{r}_1 \\
&= (x_2\hat{i} + y_2\hat{j} + z_2\hat{k}) - (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) \\
&= (x_2 - x1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k} \\
&= \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}
\end{aligned}$$

Average and Instantaneous Velocity

Average velocity
$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} = \frac{\Delta x}{\Delta t}\hat{i} + \frac{\Delta y}{\Delta t}\hat{j} + \frac{\Delta z}{\Delta t}\hat{k}$$

Instantaneous velocity
$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$
$$= v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$



The direction of the velocity is always tangent to the particle's path

Average and Instantaneous Acceleration

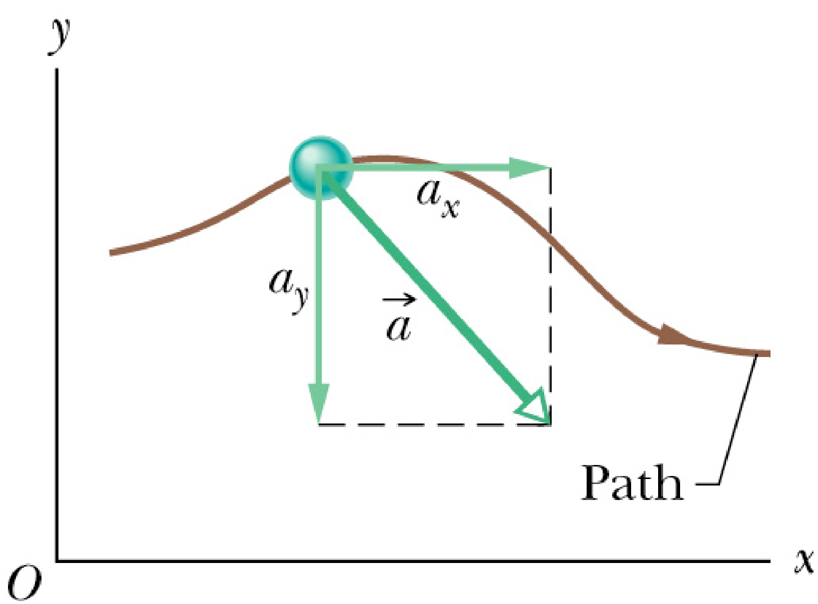
Average acceleration

$$\vec{a}_{avg} = \frac{\Delta \vec{v}}{\Delta t}$$

Instantaneous acceleration

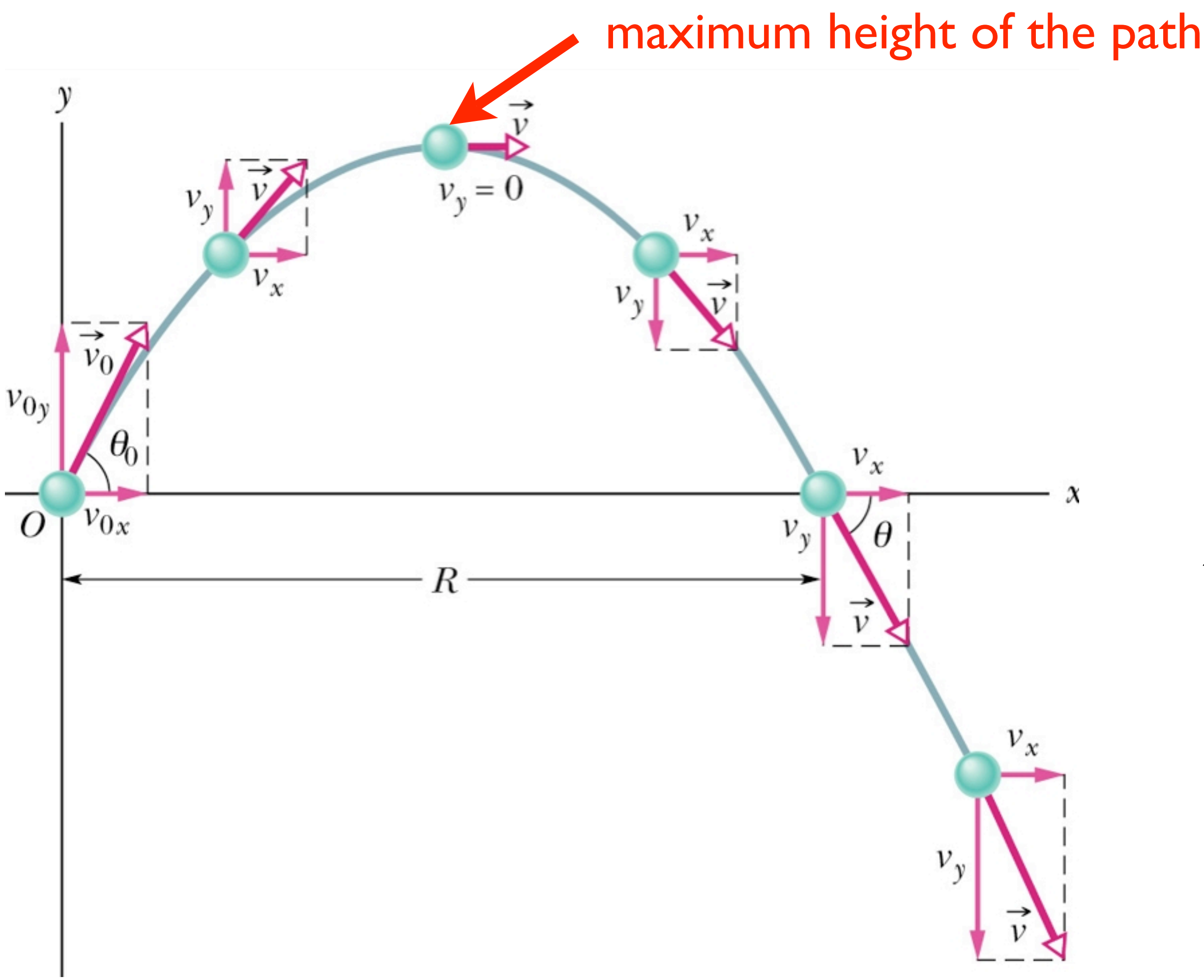
$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} + \frac{dv_z}{dt}\hat{k}$$

$$= a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$



Projectile motion

: a particle moves in a vertical plane with some initial velocity but its acceleration is always the free fall acceleration (g) then is called a projectile motion



In projectile motion, the horizontal motion and the vertical motion are independent each other

Projectile motion Analyzed

The horizontal motion: no acceleration in x direction

$$x - x_0 = v_{0x}t + \underbrace{\frac{1}{2}a_x t^2}_{=0}$$
$$= (v_0 \cos\theta_0)t$$

The vertical motion: acceleration (-g) in y direction

$$y - y_0 = v_{0y}t + \frac{1}{2}(-g)t^2$$
$$= (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

(Yes, we've learned these in ch02)

$$v = v_0 + at \quad \rightarrow \quad v_y = v_0 \sin\theta_0 - gt$$
$$v^2 = v_0^2 + 2a(x - x_0) \quad \rightarrow \quad v_y^2 = (v_0 \sin\theta_0)^2 - 2g(y - y_0)$$

Projectile motion : path

Simplicity, we let $x_0 = y_0 = 0$ (starting at the origin of the coordinate system)

$$\begin{aligned} x &= v_0 \cos\theta_0 t \\ y &= v_0 \sin\theta_0 t - \frac{1}{2} g t^2 \end{aligned}$$

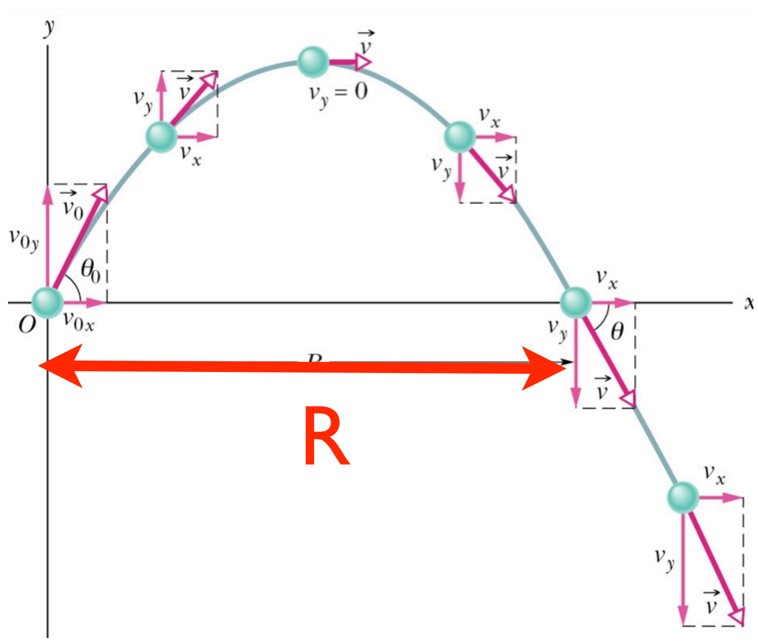
Solve the first equation for t and replace t in the 2nd equation,

$$\begin{aligned} y &= v_0 \sin\theta_0 \frac{x}{v_0 \cos\theta_0} - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0} \right)^2 \\ &= (\tan\theta_0) x - \frac{g x^2}{2 (v_0 \cos\theta_0)^2} \end{aligned}$$

This is of the form $y = ax + bx^2$, that is the equation of a parabola (포물선)

Projectile motion : horizontal range

The horizontal distance that the projectile has traveled when it returns to its initial height: R



Let x - x_0 = R and y - y_0 = 0

$$R = (v_0 \cos\theta_0)t$$

$$0 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$$

From the 2nd equation, we get

$$v_0 \sin\theta_0 = \frac{1}{2}gt$$

With the 1st equation, we eliminate t

$$v_0 \sin\theta_0 = \frac{1}{2}g\frac{R}{v_0\cos\theta_0}$$

$$R = \frac{2v_0^2}{g}\sin\theta_0\cos\theta_0$$

$$= \frac{v_0^2}{g}\sin 2\theta_0$$

$\sin 2\theta_0 = 2\sin\theta_0\cos\theta_0$ is used here

Projectile motion : horizontal range

R has its maximum value when the angle = 45°

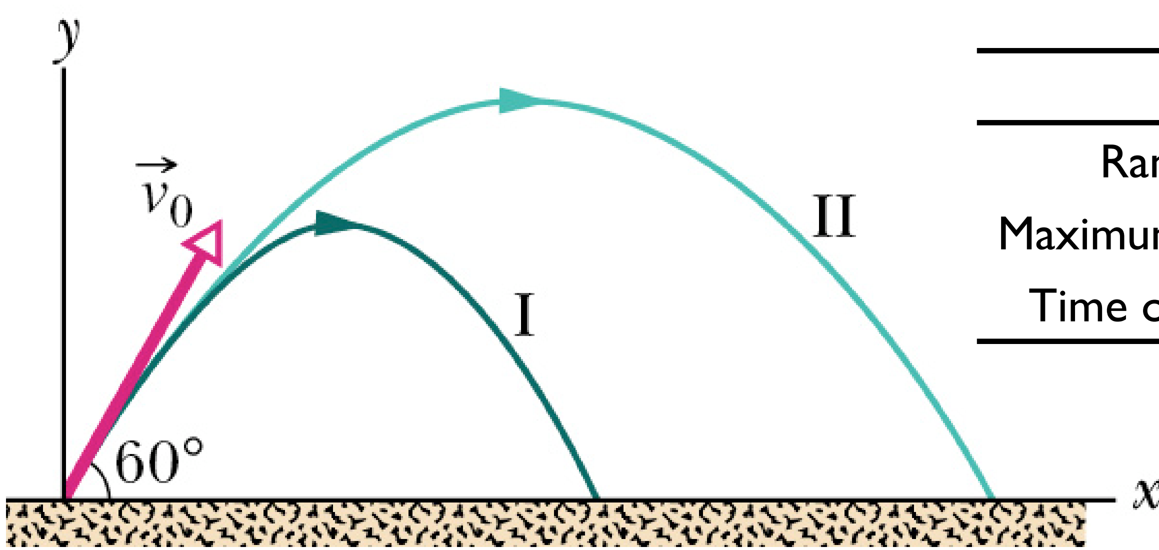
(so you should throw baseball with the angle 45° to maximize the distance?)

The effects of the air

:The air resists the motion

(angle = 60°, initial speed 44.7 m/s)

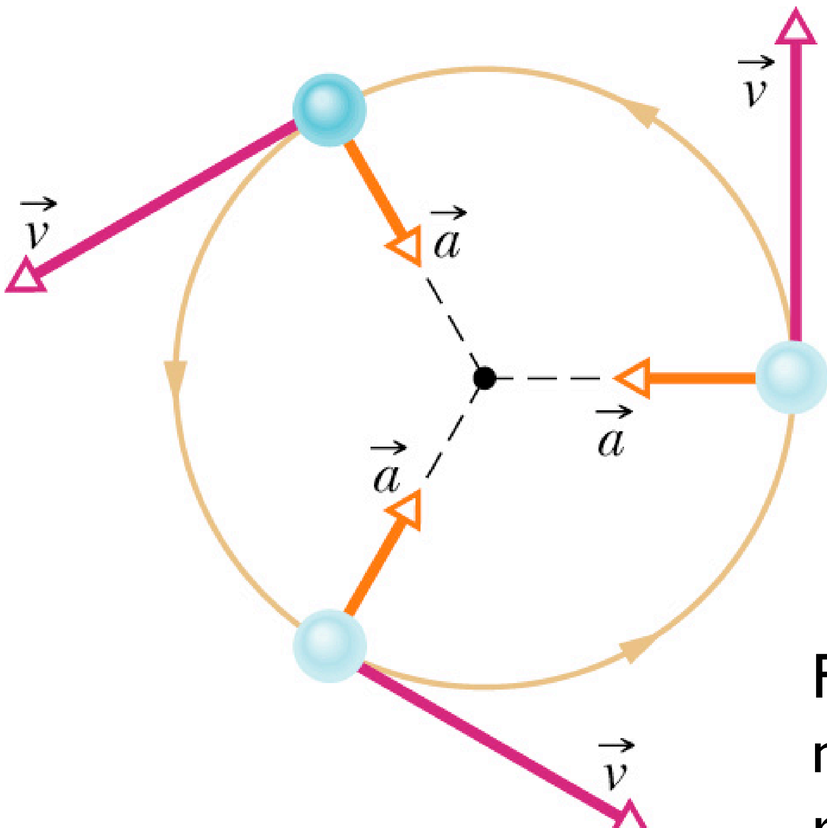
I : path with air resistance
II : path in vacuum



	Path I (air)	Path II (vacuum)
Range	98.5 m	177 m
Maximum height	53.0 m	76.8 m
Time of flight	6.6 s	7.9 s

Uniform circular motion

: if a particle travels around a circle (or a circular arc) at constant (uniform) speed, it is in uniform circular motion



speed is constant but (direction is changing) $\vec{a} = \frac{d\vec{v}}{dt} \neq 0$

$\vec{v}$ is directed tangent to the circle

$\vec{a}$ is directed radially inward

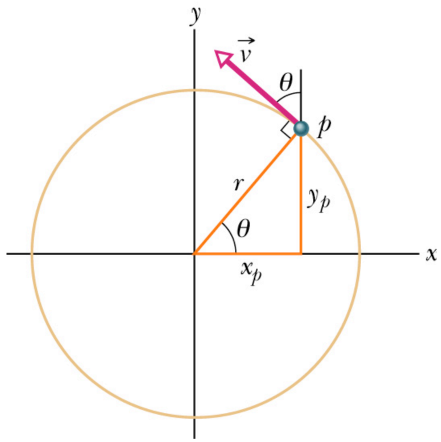
Uniform circular motion is called centripetal (center seeking, 구심성의...) acceleration

For uniform circular motion, the following relations hold (r: radius of the circle, T: period of the revolution)

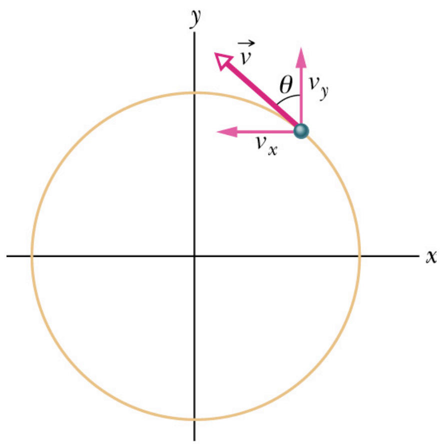
$$a = \frac{v^2}{r}$$

$$T = \frac{2\pi r}{v}$$

Uniform circular motion



(a)



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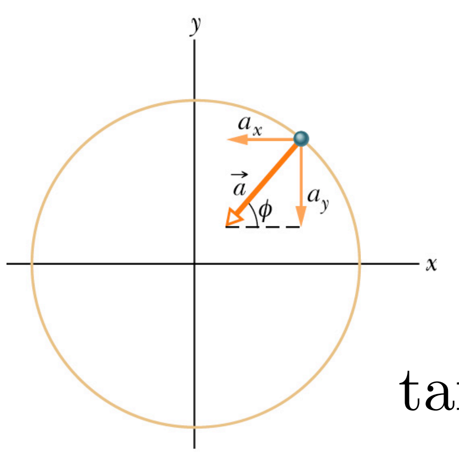


(c)

From the figure, we get

$$\begin{aligned} \vec{v} &= v_x\hat{i} + v_y\hat{j} = (-v\sin\theta)\hat{i} + (v\cos\theta)\hat{j} \\ &= (-v)\frac{y_p}{r}\hat{i} + v\frac{x_p}{r}\hat{j} \end{aligned}$$

We take the time derivative of the above:

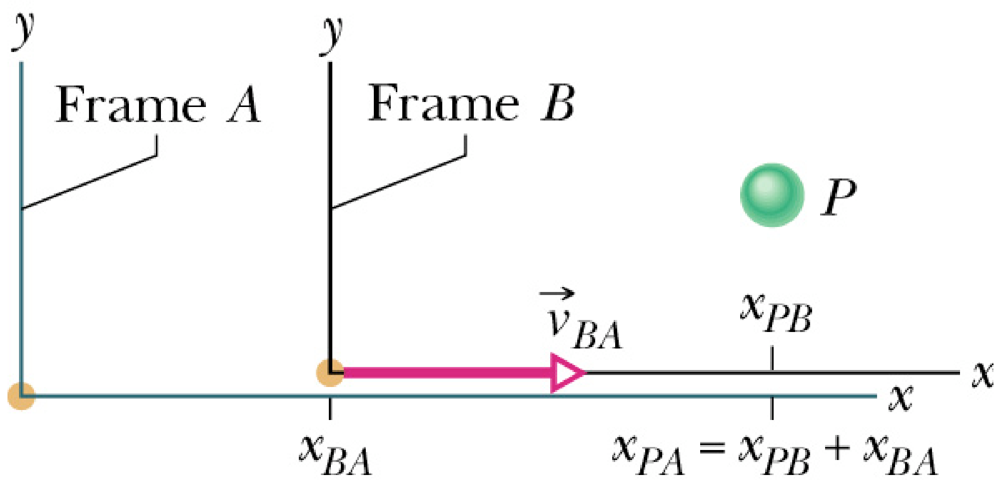
$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \left(-\frac{v}{r}\frac{dy_p}{dt}\right)\hat{i} + \left(\frac{v}{r}\frac{dx_p}{dt}\right)\hat{j} \\ &= \left(-\frac{v}{r}v_y\right)\hat{i} + \left(\frac{v}{r}v_x\right)\hat{j} \\ &= \left(-\frac{v^2}{r}\cos\theta\right)\hat{i} + \left(-\frac{v^2}{r}\sin\theta\right)\hat{j} \end{aligned}$$

$$a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r}\sqrt{\cos^2\theta + \sin^2\theta} = \frac{v^2}{r}$$

$$\tan\phi = \frac{a_y}{a_x} = \frac{-(v^2/r)\sin\theta}{-(v^2/r)\cos\theta} = \tan\theta$$

$\vec{a}$ is directed along the radius r, toward the center

Relative Motion in One dimension



The velocity of a particle depends on the reference frame

(we assume a constant speed v_{BA})

$$x_{PA} = x_{PB} + x_{BA}$$

Time derivative of the above gives

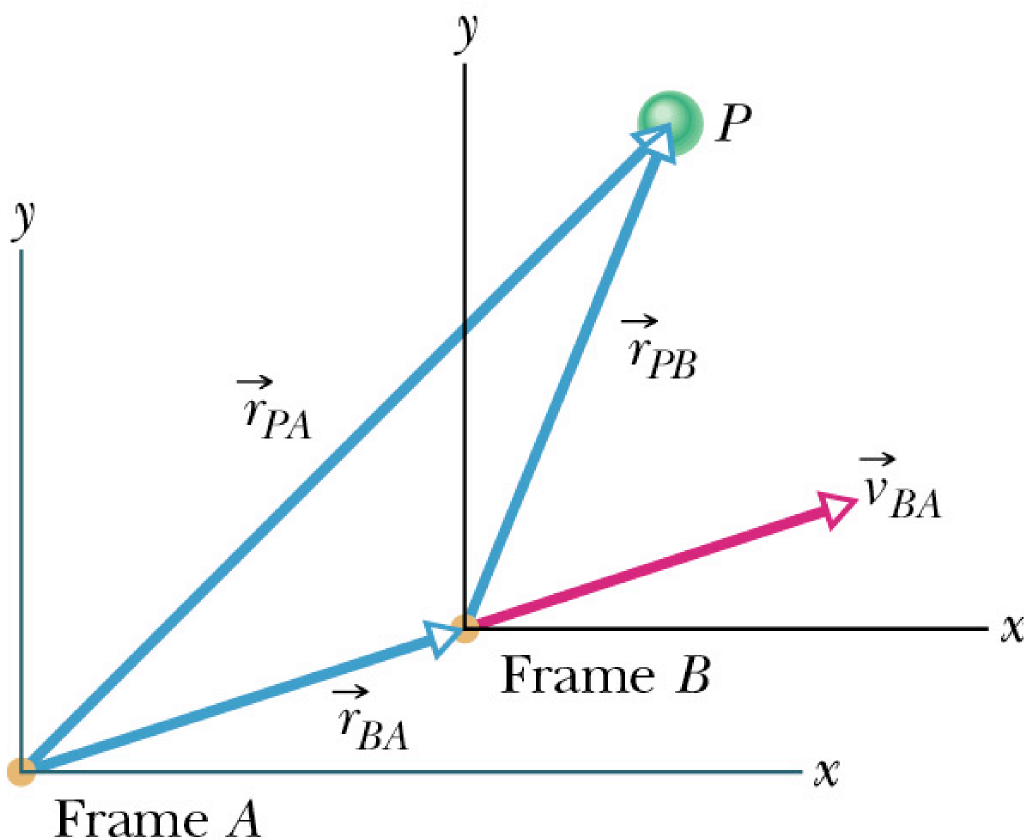
$$v_{PA} = v_{PB} + v_{BA}$$

(Special theory of relativity rejects this. We won't discuss it though)

$$a_{PA} = a_{PB}$$

$$\frac{dv_{BA}}{dt} = 0$$

Relative Motion in Two dimensions



(we assume a constant speed v_{BA})

We extend the discussion from the one dimensional case

$$\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}$$

$$\vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}$$

$$\vec{a}_{PA} = \vec{a}_{PB}$$

Summary

Position Vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Displacement Vector $\vec{\Delta r} = \vec{r}_2 - \vec{r}_1$

Projectile motion: equation of parabola

Uniform circular motion: $a = \frac{v^2}{r}$

Relative (uniform) motion: $\vec{a}_{PA} = \vec{a}_{PB}$