PHYS151 Lecture 03

Ch 03 Vectors

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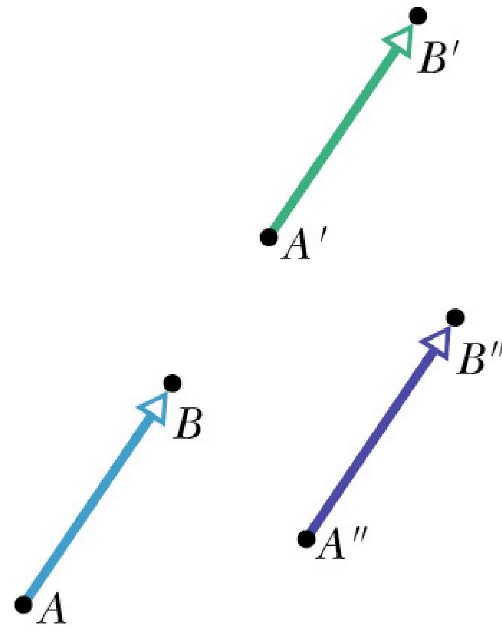
Vectors and Scalars

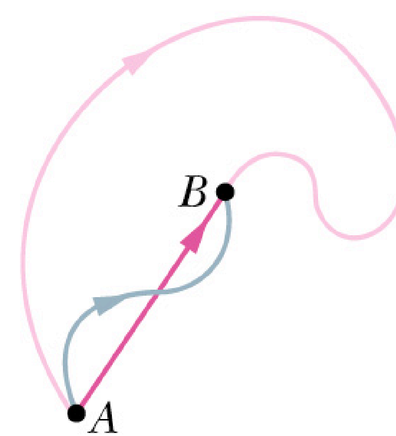
A vector has magnitude as well as direction and can be represented by an arrow (direction) with length (magnitude)
ex) Displacement vector, velocity

A scalar is a quantity does not involve direction
ex) Temperature, mass, time

a) All three arrows are same vectors
b) All three paths correspond to same displacement vector

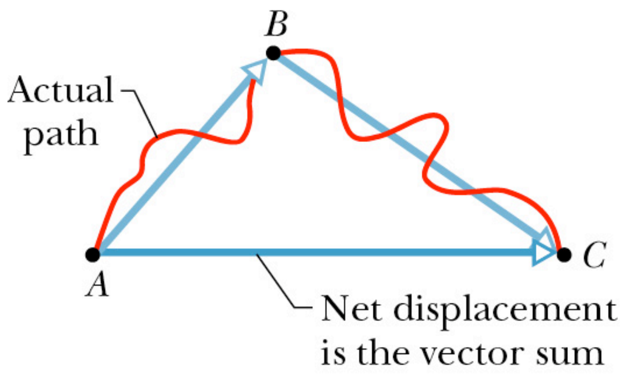


(a)



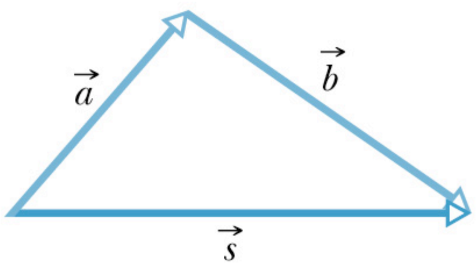
(b)

Adding Vectors Geometrically



(a)

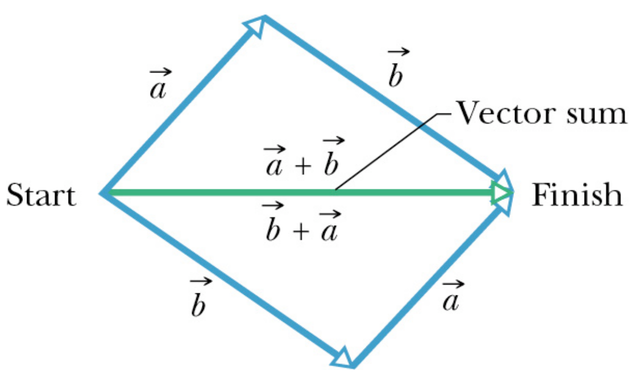
The net displacement of these two vectors is a single displacement from A to C



(b)

From the above sum, we can write

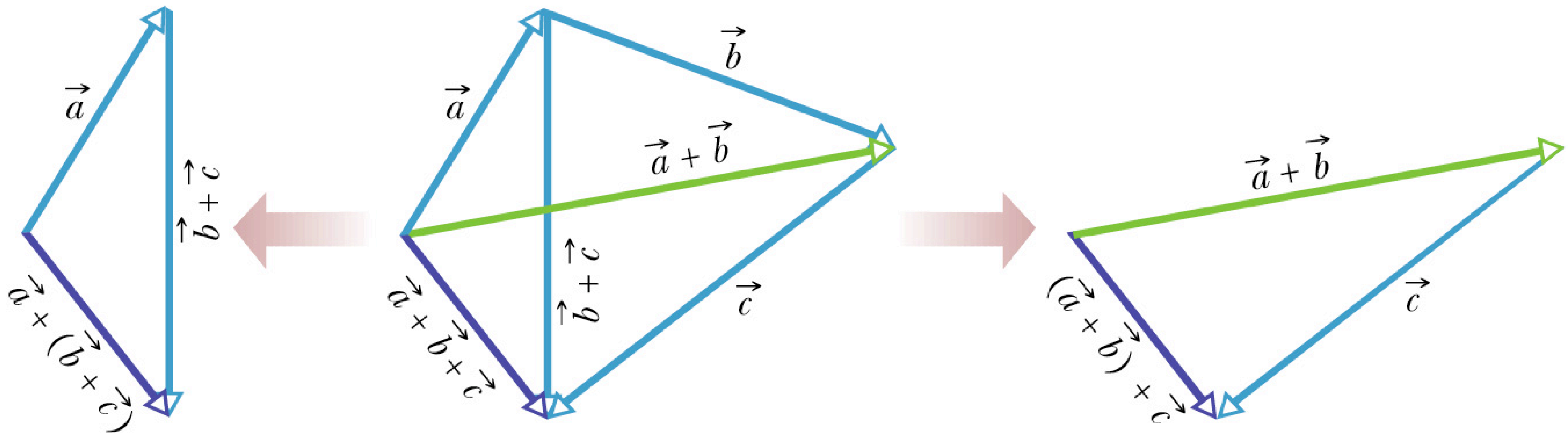
$$\vec{s} = \vec{a} + \vec{b}$$



The left figure suggests

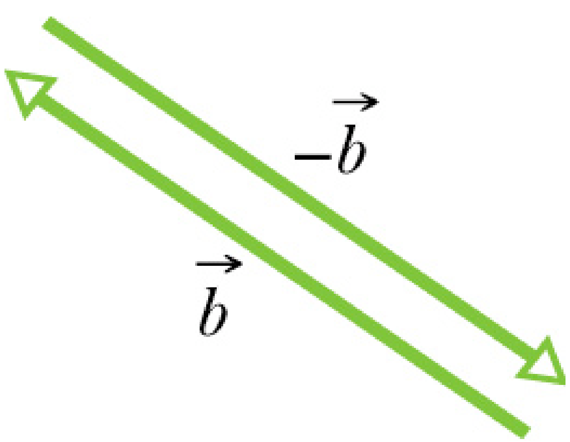
$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \quad \text{(commutative law)}$$

Adding Vectors Geometrically



From the above figure, we can write

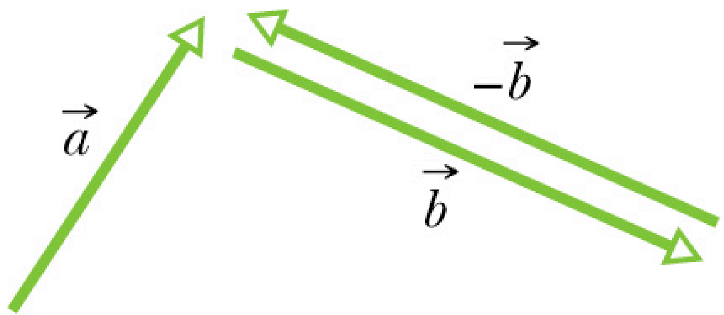
$$(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$$ (associative law)



$-\vec{b}$ is a vector with the same magnitude but the opposite direction of $\vec{b}$

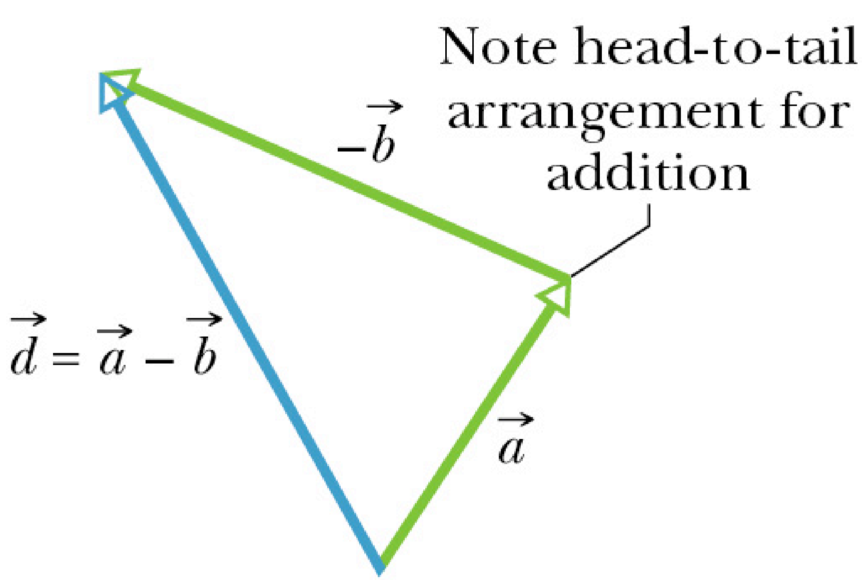
$$\vec{b}+(-\vec{b})=0$$

Adding Vectors Geometrically



(a)

Adding a negative vector has the effect of subtracting a vector



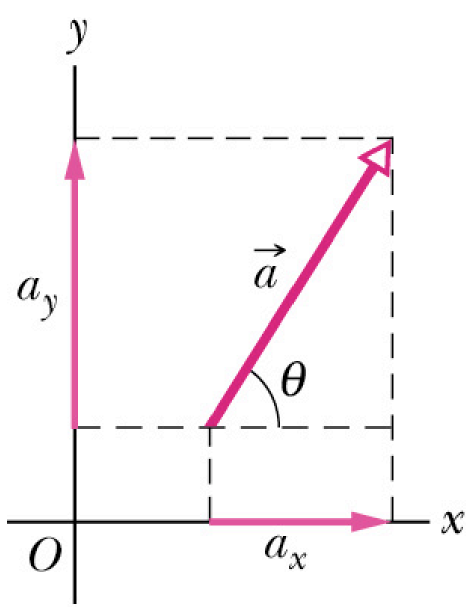
(b)

$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

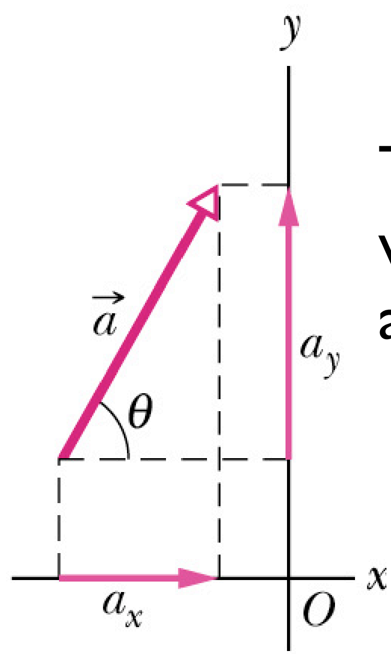
(vector subtraction)

Components of Vectors

: a component of a vector is the projection of the vector on an axis

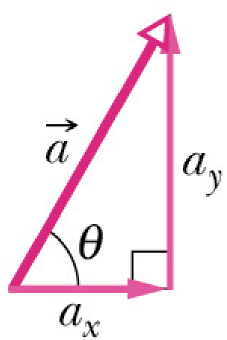


(a)



(b)

The components are unchanged if the vector is shifted (as long as the magnitude and orientation are maintained)



(c)

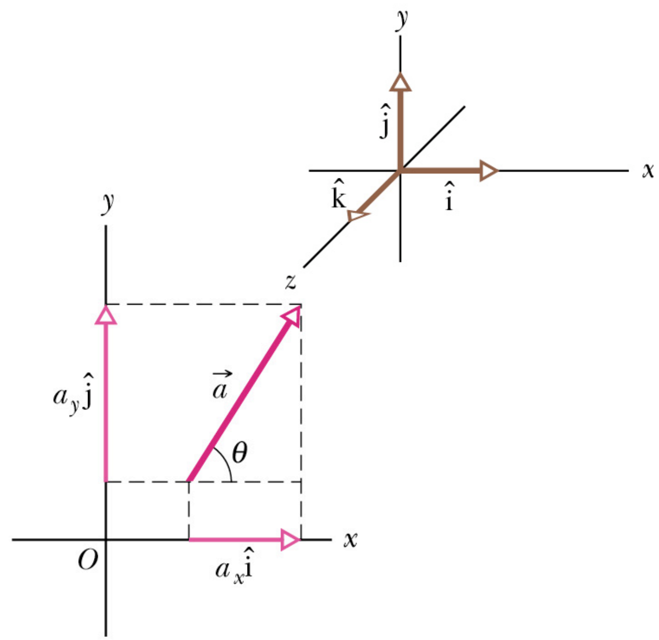
From the right triangle, we associate components with the magnitude of the vector as:

$$a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta$$

$$a = \sqrt{a_x^2 + a_y^2} \quad \text{and} \quad \tan\theta = \frac{a_y}{a_x}$$

Unit Vectors

: a unit vector is a vector that has a magnitude of exactly 1 and points in a particular direction



(a)

Unit vectors for positive directions of x,y, and z axes: $\hat{i}, \hat{j}$ and $\hat{k}$

(This arrangement of axes in figure is said to be a right-handed coordinate system)

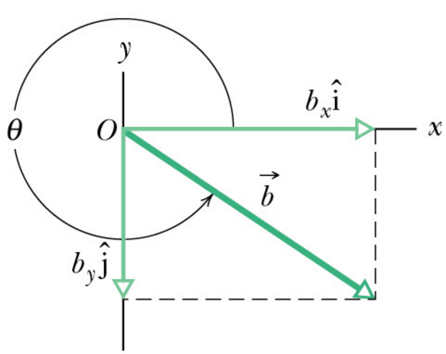
Expressing vectors by unit vectors and components:

$$\vec{a} = a_x\hat{i} + a_y\hat{j}$$

$$\vec{b} = b_x\hat{i} + b_y\hat{j}$$

$b_x\hat{i}$: vector component

b_x : scalar component



(b)

(b_y is negative for this case, see figure on left)

Adding Vectors by Components

Consider the statement : $\vec{r} = \vec{a} + \vec{b}$ This means

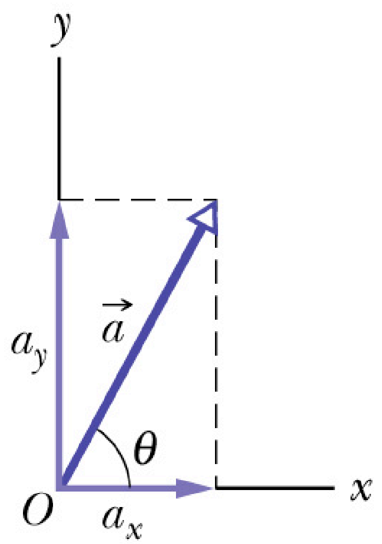
$$\begin{aligned} r_x &= a_x + b_x \\ r_y &= a_y + b_y \\ r_z &= a_z + b_z \end{aligned}$$

Similar way, we can say

$$\begin{aligned} d_x &= a_x - b_x \\ d_y &= a_y - b_y \\ d_z &= a_z - b_z \end{aligned}$$

for $\vec{d} = \vec{a} - \vec{b}$

Vectors and the Laws of Physics



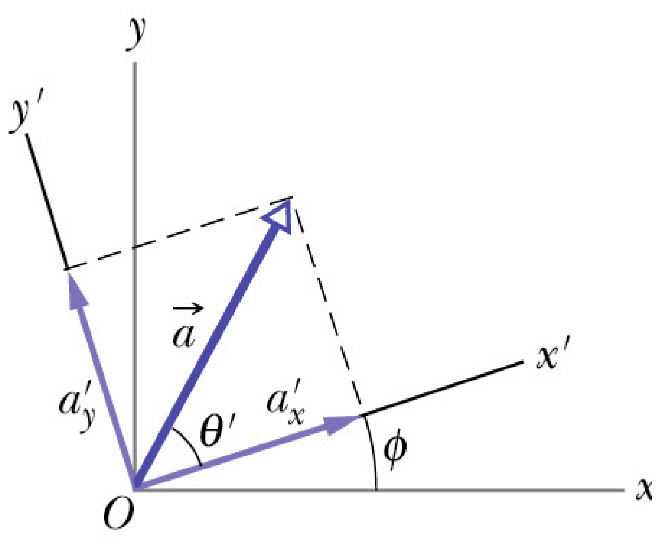
(a)

: we have freedom in choosing a coordinate system

Event if we rotate the coordinate system, the magnitude remain identical

$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a_x'^2 + a_y'^2}$$

$$\theta = \theta' + \phi$$



(b)

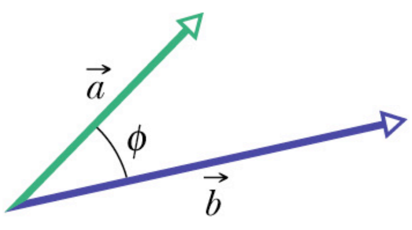
The laws of physics do not change for different coordinate system

Multiplying Vectors

Multiplying a vector by a scalar

$$s\vec{a} \quad \text{magnitude: } sa \quad \text{direction: } \hat{a}$$

Multiplying a vector by a vector

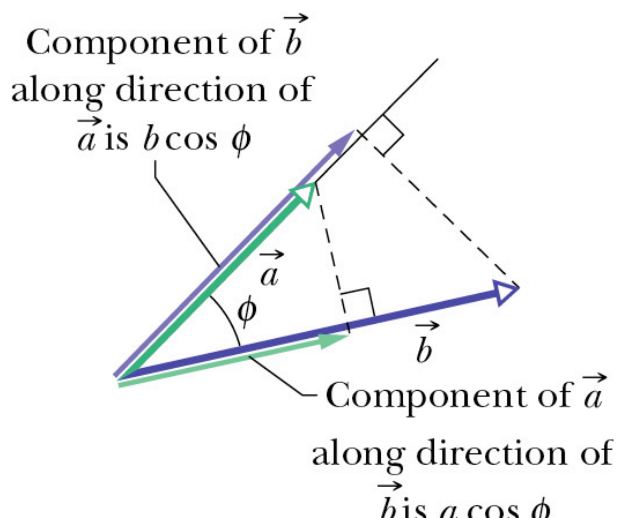


(a)

The Scalar Product (dot product)

$$\vec{a} \cdot \vec{b} = ab\cos\phi$$

ϕ : angle between directions of two vectors

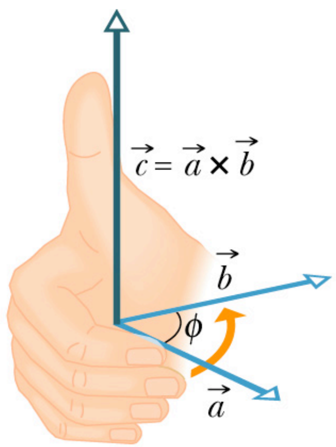


(b)

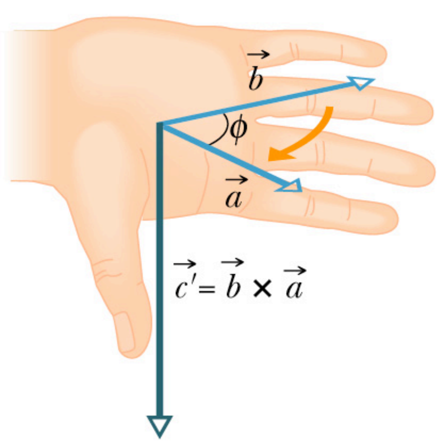
$$\begin{aligned}\vec{a} \cdot \vec{b} &= (a_x\hat{i} + a_y\hat{j} + a_z\hat{k}) \cdot (b_x\hat{i} + b_y\hat{j} + b_z\hat{k}) \\ &= a_x b_x + a_y b_y + a_z b_z\end{aligned}$$

Multiplying Vectors

Multiplying a vector by a vector



(*a*)



(*b*)

The Vector Product (cross product)

$$\vec{c} = \vec{a} \times \vec{b}$$

magnitude: $c = ab \sin \phi$

ϕ : the smaller angle between two vectors

direction: from the right-hand rule

note: $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

$$\begin{aligned} \hat{i} \times \hat{i} &= 0 \\ \hat{i} \times \hat{j} &= \hat{k} \\ \hat{i} \times \hat{k} &= -\hat{j} \\ \hat{j} \times \hat{k} &= \hat{i} \end{aligned}$$

$$\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}$$

Multiplying Vectors

Cyclic properties (advanced):

For the vector products of unit vectors, we have:

$$\begin{aligned} \hat{i} \times \hat{i} &= 0 \\ \hat{i} \times \hat{j} &= \hat{k} \\ \hat{i} \times \hat{k} &= -\hat{j} \\ \hat{j} \times \hat{k} &= \hat{i} \end{aligned}$$

We can remember this as follows:

1) Assign 1,2,3 for $\vec{1} \times \vec{2} = \vec{3}$

2) There are 6 different ways to permutate three numbers (3!=6, see below)

1,2,3
1,3,2
2,1,3
2,3,1
3,1,2
3,2,1

Blue ones are same if we move numbers cyclic manner (2,3,1 => 3,1,2 => 1,2,3: we move the first number from the head to tail...) : have positive sign for the resulting vector product

Red ones are also same (1,3,2 => 3,2,1 => 2,1,3) : have negative sign

Summary

Components of a Vector: $a_x = a\cos\theta \quad \text{and} \quad a_y = a\sin\theta$

Unit-vector notation: $\vec{a} = a_x\hat{i} + a_y\hat{j}$ (2-dim. case)

Adding vectors: $\vec{r} = \vec{a} + \vec{b}$

$$\begin{aligned} r_x &= a_x + b_x \\ r_y &= a_y + b_y \\ r_z &= a_z + b_z \end{aligned}$$

The Scalar Product: $\vec{a}\cdot\vec{b} = ab\cos\phi$

The Vector Product: $\vec{c} = \vec{a}\times\vec{b}$

$$\vec{a}\times\vec{b} = (a_y b_z - b_y a_z)\hat{i} + (a_z b_x - b_z a_x)\hat{j} + (a_x b_y - b_x a_y)\hat{k}$$